INTERNATIONAL JOURNAL OF ABSTRACTS

STATISTICAL THEORY AND METHOD

VOLUME 2 · NUMBER 3

1961

PUBLISHED FOR
THE INTERNATIONAL STATISTICAL INSTITUTE
OLIVER AND BOYD

VOLUME 2 · NUMBER 3

Abbreviation (*World List* Style):
Int. J. Abstr., Statist. Theory

Short Title:
Statistical Theory Abstracts

PRINTED IN GREAT BRITAIN BY
OLIVER AND BOYD LTD., EDINBURGH

INTERNATIONAL JOURNAL OF ABSTRACTS STATISTICAL THEORY AND METHOD

COVERAGE OF JOURNAL

THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which report only applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. A complete list of journals covered is printed in the annual Index Supplement. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted :

Annals of Mathematical Statistics
Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

Within the larger group of journals, which are not wholly devoted to statistical theory and method, there are some journals which have the vast majority of their contributions in this field. These journals, therefore, will be abstracted on a virtually complete basis :

Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Sankhyā

After experience in the publication of this journal of abstracts it may be found desirable to add further journals to these lists. In any case readers may be assured that all papers properly to be included in this journal of abstracts will be included irrespective of such notification. If any reader of this journal discovers a paper which happens to have been overlooked, the General Editor will be pleased to be informed so that the appropriate abstract can be made : always provided that the date of publication is after 1st October 1958, when the abstracting for this journal commenced.

In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

FORMAT OF JOURNAL

The abstracts will be up to 400 words long—the recommendation of UNESCO for the " long " abstract service : they will be in the English language although the language of the original paper will be indicated on the abstract together with the name of the abstractor. In addition, the address of the author(s) will be given in sufficient detail to facilitate contact in order to obtain further detail or request an offprint. Suitable indexes will be provided annually in a supplement.

The scheme of classification has been developed upon lines that will facilitate the transfer to punched cards of the code numbers allocated to each abstract: to allow for future development it is suggested that use is made of 4-column fields. Each abstract will have two classification numbers: the primary number in heavy type to indicate the basic topic of the paper and the secondary number in brackets to take account of the most important cross-reference. The sheets of the journal are colour-coded according to the twelve main sections of the classification and it should be noted that it is the main section number of the primary classification which determines the colour code for each abstract. It is believed that this method of colour-coding the pages will provide a distinctive visual aid to the identification of abstracts both when the journal is in bound form or dismantled and filed on cards or in binders. The format and simple binding allows of the following alternative treatments by users of the journal:

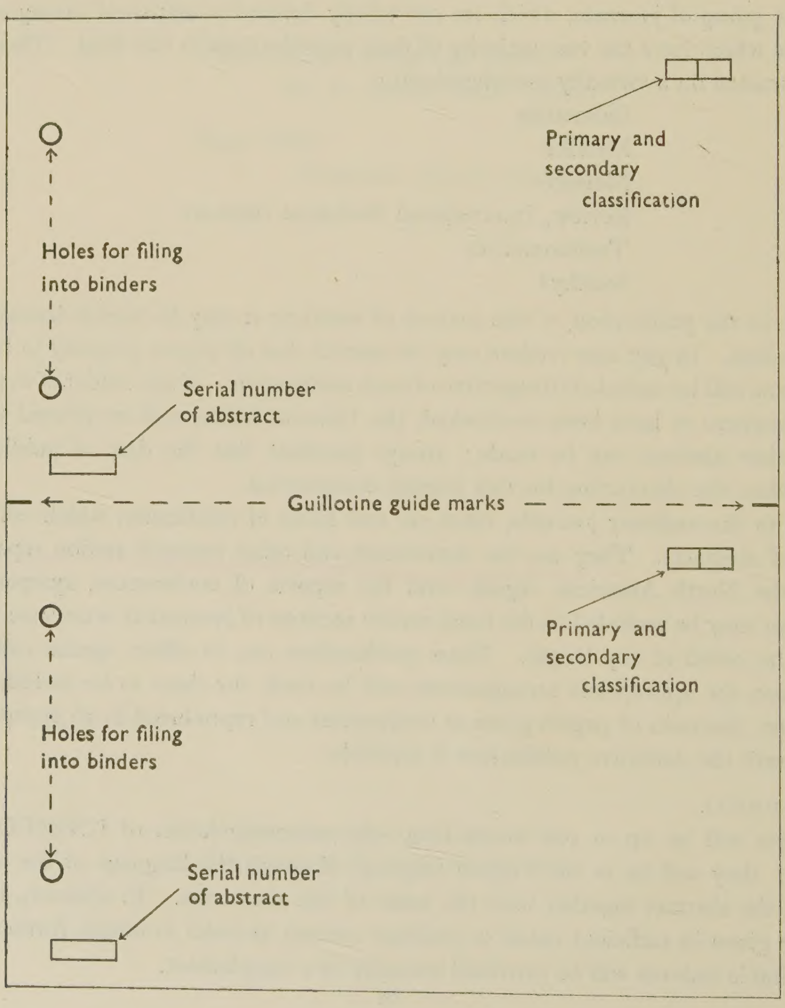
(*a*) Leave intact as a shelf-periodical.

(*b*) Split and filed in page form according to the main sections of the classification.

(*c*) Split and guillotine (single cut) each page ready for:

- (i) pasting onto standard index cards.
- (ii) filing in loose-leaf or other binders—for which the appropriate holes are punched.

It will also be possible for those users who need a completely referenced file to insert skeleton cards or sheets according to the secondary classification number provided on each abstract.



JOURNAL OF THE ROYAL STATISTICAL SOCIETY

SERIES A (GENERAL), four issues a year, 25s. per part, annual subscription £4. 4s. (U.S. $12) post free; SERIES B (METHODOLOGICAL), two issues a year, 40s. per part, annual subscription £3. 2s. (U.S. $9) post free.

SERIES A (GENERAL)

VOL. 124, PART 2, 1961

SERIES B (METHODOLOGICAL)

VOL. 23, No. 1, 1961

ROYAL STATISTICAL SOCIETY, 21 BENTINCK STREET, LONDON, W.1

THE ANNALS OF MATHEMATICAL STATISTICS

Vol. 32, No. 3—September 1961

CONTENTS

BIOMETRIKA

Volume 48, Parts 1 and 2 **CONTENTS** June 1961

The subscription, *payable in advance*, is 54/- (or $8.00) per volume (including postage). Cheques should be made payable to *Biometrika*, crossed " a/c *Biometrika Trust* " and sent to The Secretary, Biometrika Office, University College, London, W.C.1. All foreign cheques must be drawn on a Bank having a London agency.

Issued by THE BIOMETRIKA OFFICE, University College, London

BIOMETRICS

Journal of the Biometric Society

VOL. 17, No. 3 SEPTEMBER 1961

CONTENTS

Biometrics is published quarterly. Its objects are to describe and exemplify the use of mathematical and statistical methods in biological and related sciences, in a form assimilable by experimenters. The annual non-member subscription rate is $7. Inquiries, orders for back issues and non-member subscriptions should be addressed to:

BIOMETRICS, Department of Statistics, The Florida State University, Tallahassee, Florida

AUTHOR INDEX

VOL. 2: No. 3 (427-598)

Volume 1. CORRECTIONS

1/329	**0.8**	Krelle	Volume number of journal abstracted should read, **2**.
1/558	**1.0**	Theodorescu	,, ,, ,, ,, ,, ,, ,, **2**.

Volume 2. CORRECTIONS

2/25	**2.1**	Filippo	Author's name should read, Emannelli, F.
2/289	**4.0**	Czen & Dzan	Language of original paper was *Polish*.

AKAIKE, H. & SAIGUSA, Y. (Institute of Statistical Mathematics, Tokyo) **0.6** (0.1)

On a min-max theorem and some of its applications—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 1-5 (1 reference)

In this paper the authors show that for a matrix with elements arranged in decreasing order in each row, we have only to select successively the rows with the smallest diagonal elements in order to rearrange the rows of the matrix, so that we obtain the minimum of the maximum diagonal elements of the rearranged matrices; that is, we have only to take as the new jth row the one with the smallest jth element in the remaining rows.

Two examples of the application of this result are given:

(i) application to the analysis of rational economic behaviour of a consumer who tries to keep minimum the maximum expenses in buying successively a set of commodities with gradually decreasing prices

(ii) preparation to assure a good convergence rate in applying Seidel's process of successive approximation for the solution of simultaneous linear equations.

(H. Akaike)

2/427

DANZIG, G. B. (RAND Corp., Santa Monica, Cal.) **0.8** (9.1)

On the significance of solving linear programming problems with some integer variables—*In English*

Econometrica (1960) **28**, 30-42 (10 references)

This paper deals with the consideration that recent proposals by Gomory [" Essentials of an algorithm for integer solutions to linear programmes," *Bull. Amer. Math. Soc.* (1958) **64**] and by Beale [" A method of solving linear programming problems when some but not all of the variables must take integer values," *Statist. Tech. Res. Group* (1958) Princeton] for solving linear programmes involving integer-valued variables appear sufficiently promising that it is worthwhile to systematically review and classify problems that can be reduced to this class and thereby solved.

The author commences by discussing the " cutting plane approach " which was first proposed and demonstrated by Fulkerson, Johnson & Danzig, G. B. [" Solution of a large scale travelling salesman problem," *J. Operat. Res. Soc.* (1954) **4**, 393-410] and further explored by Manne & Markowitz [" On the solution of discrete programming problems," *Econometrica* (1957) **25**, 19]. Gomory developed a theory of automatic generating " cutting planes " and the theory was generalised to the case where some variables are constrained and some continuous by Beale.

The second section deals with the fixed charge problem and is followed by the application of the principles to the orthogonal latin-square problem: in this context the author remarks on the fact that Bose & Shrikhande recently proved that Euler's famous conjecture about the non-existence of orthogonal latin-squares of certain even orders was false [*Notices of the American Math. Soc.* (1959); abstract No. 558-27].

The fifth section discusses the classic problem of colouring a map using at most four colours so that no two regions having a boundary in common have the same colour, but the author states that " The two ways that are given to colour constructively a particular map do not contribute anything to a proof or the truth or falsity of the conjecture except that an efficient way for solving particular problems on an electronic computer may provide a counter example ".

Historically, non-linear, non-convex and combinatorial problems are areas where classical mathematics almost always fail and it is therefore significant that a reduction to class of linear programming involving integer-valued variables can be made.

(W. R. Buckland)

2/428

FERRERI, C. (School of Statistics, Palermo) **0.2** (–.–)

On a mathematical theory of the survival and the fit of some new functions to the Sicilian population—*In Italian*

Ann. Fac. Econ. Com. Palermo (1958) **12**, 1-70 (18 references, 13 tables, 5 figures)

Quiquet's theory on survival functions is generalised. The author shows the practical impossibility of choosing rationally a function representing the survivals for all ages of a population. He later supposes a subdivision of the survivor's distribution into the following three periods: (*a*) childhood, (*b*) middle age, (*c*) old age. He assumes, as discriminant values, the years corresponding to the minimum and to maximum of death.

The author shows that the Lazarus function may describe the middle age better than many other functions; for childhood and old age he proposes, respectively, the new functions

$$l(x) = As^x g^{x^3} b^{x^\alpha} \quad \text{and} \quad l(x) = A(\omega - x)^\alpha b^x$$

where the parameters are posed with regard to biological, sociological and demographic knowledge of human mortality.

After some critical remarks on Vincent's criterion determining the oldest age, the author using the mean-square method fits his functions to the survival distribution of the Sicilian population. He gets a new death table in general agreement with the empirical date.

(F. Galantino)

2/429

FORTET, R. (University of Paris) **0.0** (0.8)

The algebra of Boole and its applications in operational research—*In French*

Trab. Estadíst. (1960) **11**, 111-118 (9 references)

Boolean algebra is defined as having two elements, say 0 and 1. Boolean addition and multiplication are defined. Boolean functions are defined as functions of Boolean variables, that is, variables which can only take values 0 and 1. Boolean equations are defined as $f(x_1, \ldots, x_n) = 0$, where f is a Boolean function and x_i a Boolean variable. A set of equations of n Boolean variables is obtained as a solution for a condition imposed on the n Boolean variables. A method to solve systematically a condition C expressed by an equation of the type $f(x_1, \ldots, x_n) = 0$ is given.

A function $f(x_1, \ldots, x_n)$ of n Boolean variables is said to be a whole algebraic function if it can be expressed as a polynomial in the sense of the ordinary algebra on the x_i and x_j. A whole algebraic function can always be transformed in a linear function in the sense of the ordinary algebra if convenient supplementary variables and conditions with whole coefficients in the initial and supplementary variables are introduced.

An algebraic Boolean programme is defined as follows: if $x_1, \ldots, x_n$ are Boolean variables for which a set of conditions R holds and $f(x_1, \ldots, x_n)$ is a whole algebraic function of the x_i, we have to find the elements of the set $\mathscr{V}_{(n)}$ in which the x_i are defined that maximise f and satisfy R. Boolean programmes are very frequent in nature but they are very difficult to solve. A systematic method of proving all the 2^n elements of $\mathscr{V}_{(n)}$ is given in order to solve a Boolean programme. An example is studied. Equalities and inequalities for whole algebraic functions are studied. For these we have the following theorem: every algebraic Boolean programme where the condition R is expressed by algebraic equations or inequations can be carried to a system of whole algebraic equations or inequations. It is possible to solve a whole algebraic equation or inequation by carrying it to an algebraic Boolean equation, at least when the numerical coefficients are whole numbers; this is enough for practical applications. Two examples are studied.

(M. Melendez)

2/430

GREVILLE, T. N. E. (Office of the Quartermaster General, Washington) **0.6** (6.1)

Some applications of the pseudo-inverse of a matrix—*In English*

SIAM Rev. (1960) **2**, 15-22 (13 references)

An $m \times n$ matrix A of rank $r>0$ can be expressed as a product $A = BC$ where B is an $m \times r$ matrix and C is an $r \times n$ matrix, and both are of rank r. Then the pseudo-inverse of A is given by $A^P = C^T(CC^T)^{-1}(B^TB)^{-1}B^T$ where the superscript T denotes transpose. The author describes certain advantages in using the pseudo-inverse for obtaining orthogonal polynomials over a discrete domain. He also derives a recursive algorithm for the pseudo-inverse which is useful in fitting polynomials of successively higher degree. Attention is drawn to the fact that " pseudo-inverse " is the same as the " conditional inverse " as used by Bose (Department of Statistics, University of North Carolina). This idea has widespread application in the general theory of linear estimation.

(W. T. Wells)

2/431

HANSON, M. A. (University of New South Wales, Sydney) **0.8** (7.5)

Errors and stochastic variations in linear programming—*In English*

Aust. J. Statist. (1960) **2**, 41-46 (2 references)

This paper points out that small errors in the data of a linear programme may cause a large variation in the desired maximum. Such errors should therefore be closely investigated, as their practical effects may outweigh the advantages of linear programming.

Problems of linear programming amount to determining an n-dimensional vector $\boldsymbol{x}$ (with non-negative elements) subject to the condition $A\boldsymbol{x} = \boldsymbol{b}$ where A is an $m \times n$ matrix and $\boldsymbol{b}$ an m-dimensional vector, in such a way that the scalar product $\boldsymbol{cx}$ is maximised.

The effects of fluctuation in the coefficients of A, $\boldsymbol{b}$, $\boldsymbol{c}$, and of the wrong choice of basic variables on the maximum of $\boldsymbol{cx}$ are examined.

A simple example is given in which the coefficients of A alone are subject to error; it is shown that a coefficient of variation of two per cent. in these causes an increase of 50 per cent. in the maximum. The author concludes " that one should not accept the results of linear programming without an analysis of the errors involved ".

(J. Gani)

2/432

JIŘINA, M. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **0.1** (10.6)

Ordinary differential equations with random coefficients and random right-hand side—*In English*

Second Hungarian Mathematical Congress (1961)

The author is concerned with the differential equation

$$\sum_{m=0}^{r} c_m x^{(m)} = y.$$

We shall suppose that c_m are complex-valued random variables. The smallest σ-algebra generated by them will be denoted by τ. A wide-sense stationary random distribution will be called τ-stationary, after Ito, if the wide-sense stationarity condition is required for conditional expectations with respect to τ instead of ordinary expectations. Under the condition that y is a τ-stationary random distribution, the necessary and sufficient condition for the existence of a τ-stationary random distribution x which satisfies the equation is presented.

The general form of x is described by means of its spectral decomposition. This makes possible the description of the general form of the corresponding correlation distribution and spectral measure.

(M. Jiřina)

2/433

KNÖDEL, W. (Technical High School, Vienna) **0.8** (-.-)

A generalised transportation problem—*In German*

Unternehmensforschung (1960) **4**, 1-12 (5 references)

A generalised transportation problem is treated in which each supplier has the possibility of producing a number of different products. An optimum solution minimising total cost must give the optimum shipment plan and state at the same time how the production capacity of each supplier is used for the different products. This problem can be treated as a linear programming problem and solved by the simplex technique. To avoid the computational burden, it is shown how this problem can be reduced to the ordinary transportation problem which can be solved by the stepping-stone method. This can be done by the two methods of introducing fictional transportation costs and localities.

In the first case, the production of different commodities by a single factory is treated as different supply points. Similarly the points of destination are dealt with. No delivery x_{uv} from supply point P_u to a destination B_v is allowed, thus $x_{uv} = 0$, if $u \neq v$. If the stepping-stone method is applied, it seems rather difficult to take account of this condition in the actual calculations. However, if the corresponding unit transportation cost a_{uv} is selected as $a_{uv} = \infty$ or, to allow numerical calculations, is given a sufficiently high finite value, this route will not appear in the optimal solution thus fulfilling the above condition. In the second case, if total production capacity surmounts total demand, inequalities appear. These can be converted into equalities by introducing slack variables. This can be interpreted as adding a fictive point of destination B_0 to which the amounts of goods not produced are being shipped at zero transportation costs. The total amount B_0 equals total production capacity minus total demand.

The method is applied to two special types of problems: in the first one each supplier can produce any combination of different products within his capacity limits. In the second one, it is assumed that each supplier can produce any one of the commodities, but only one at a given time. Finally the more general case is treated, in this case a producer is only willing to devote part of his capacity to products of comparatively low profit margins. A formal proof for this method is given, this can also be applied to similar linear programming problems of a more general type.

(H. Gülicher)

2/434

PATTERSON, H. D. (Rothamsted Experimental Station) **0.2** (4.3)

A further note on a simple method for fitting an exponential curve—*In English*

Biometrika (1960) **47**, 177-180 (5 references)

The author is concerned with the fitting of the exponential regression curve $\alpha - \beta\rho^x (0 < \rho < 1)$ which he has discussed previously [" A simple method for fitting an asymptotic regression curve " *Biometrics* (1956) **12**, 323-329 and later Patterson & Lipton " An investigation of Hartley's method for fitting an exponential curve " *Biometrika* (1959) **46**, 281-292; abstracted in this journal No. **1**/602, **4.3**].

In this paper he presents a modification of his approximate method which leads to increased efficiency without adding considerably to the arithmetic. Up to twelve equally spaced values of x are considered, and the estimation of e involves simply the root of a quadratic equation whose coefficients are given by linear functions of the ordinates. It is suggested that unless ρ is very small and the number of values of x very large this method will give estimates of the parameters which are of high efficiency and which are close to the least-squares estimates. Numerical examples are given.

(Florence N. David)

2/435

PIEHLER, J. (Leuna-Werke, Merseburg, Germany) **0.8** (-.-)

A note on sequencing problems—*In German*

Unternehmensforschung (1960) **4**, 138-142 (3 references)

Under certain restrictions the sequencing problem for m jobs and n machines is reduced to the travelling-salesman problem. These problems are concerned with situations where each of m jobs requires time on some or all of n different machines. The times required by the m jobs differ from machine to machine.

The problem is to determine a sequence of the m jobs; a permutation of the integers 1, 2, ..., m, which minimises the total elapsed time. In the following case it is supposed that all jobs require time on each of the machines and that the sequence of machines must be the same for all jobs. Later it is supposed that there are no possibilities of intermediate storing.

Let a_{ik} be the time required by job i on machine k. Then the total elapsed time is given by

$$Z = \sum_{k=1}^{m} a_{mk} + \sum_{i=1}^{m-1} \max_{1 \leq k \leq m} \left\{ \sum_{\nu=1}^{k} a_{i\nu} - \sum_{\nu=1}^{k-1} a_{i+1,\nu} \right\}.$$

It is to be noted that Z is to be evaluated for any sequence of jobs. In order to minimise Z, or to find the sequence minimising Z, one must determine the matrix

$$A_{ij} = \begin{cases} \max\limits_{1 \leq k \leq m} \left\{ \sum\limits_{\nu=1}^{k} a_{i\nu} - \sum\limits_{\nu=1}^{k-1} a_{j\nu} \right\} & \text{for } j \neq i \\ M & \text{for } j = i \end{cases}$$

and solve m travelling-salesman problems, given by the matrix A_{ij}, so that in each of these travelling-salesman problems one of the m jobs is treated as the last one in the sequence.

A numerical example illustrates the method presented.

(W. Dinkelbach)

2/436

VERHAGEN, A. M. W. (Commonw. Sci. Industr. Res. Organisation, Melbourne) **0.2** (10.9)

Growth curves and their functional form—*In English*

Aust. J. Statist. (1960) **2**, 122-127 (10 references)

The paper starts with an *a priori* justification and introduction of Gompertz and logistic curves in dealing with growth data, and makes a comparison of the shapes of these curves. A close approximation may be obtained by matching the branches of the Gompertz and logistic curves on either side of the inflection point by exponential functions. Functional transformations can be made on the Gompertz and logistic functions and the three parameters estimated. The success of the Gompertz and logistic curves as graduation formulae, despite operational difficulties, appears due to the fact that, irrespective of the parameter values, they have the sigmoid shape characteristic of most growth processes in addition to having three adjustable parameters available for the matching of the observations. The Gompertz curve and the logistic curve are discussed in relation to the allometric and linear transformations.

The rest of the paper is devoted to the investigation of the general functional form $rH(p+qt)$ in relation to these transformations. It is established that the class of exponential functions $\beta \exp(\gamma t)$ is the only one with the general form $rH(p+qt)$ such that any two members are related by an allometric transformation. The exponential function does not lose its form under any allometric transformation. The Gompertz does not lose its form if the constant term μ, is zero.

(A. M. W. Verhagen)

2/437

WOLFE, P. **0.8** (–.–)

The simplex method for quadratic programming—*In English*

Econometrica (1959) **27**, 382-398 (9 references, 2 figures)

After defining " quadratic programming " the author states four problems which are of some interest: regression, efficient production, the " portfolio " problem, and convex programming. The main difference between the method proposed by the author and other methods for the computational solution of quadratic programming problems lies in his use of the simplex method for linear programming. This is developed in two forms: the " short " and the " long " method, the latter arising on solving the following quadratic problem for all $\lambda \geqq 0$:

$$\text{minimise } f(\lambda, \boldsymbol{x}) = \lambda \boldsymbol{p}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'C\boldsymbol{x}$$

where $x \geqq 0$ and $Ax = b$: C is an n by n symmetric matrix, and $\boldsymbol{p}$ a one by n matrix.

After discussing some preliminary points the author outlines the computation in section three; followed by an example in both the forms mentioned above. Section five contains the relevant proofs for the reversions given in section three and the final section modifies the computations for other types of constraints.

(W. R. Buckland)

2/438

CLUNIES-ROSS, C. W. & RIFFENBURGH, R. H. (Virginia Poly., Blacksburg, and Univ. Hawaii) **1.4** (6.4)

Geometry and linear discrimination—*In English*
Biometrika (1960) **47**, 185-189 (1 reference)

This paper is concerned with the problem of classifying an individual into one of two multivariate populations. The discriminator is minimax and is assumed to be linear in the measurements; that is to say it is assumed to be a hyperplane in the multidimensional space.

It is shown that provided an equation (authors' 4.1) has a negative root, the optimum discriminant hyperplane passes between the centres of the two populations, that is to say that the optimum discriminator discriminates between the means of the two populations.

When the equation (4.1) does not have a negative root there is no clear-cut result. The best discriminating hyperplane will not pass between the population centres and the minimax procedure favours the population with the greatest weight factor.

(Florence N. David)

2/439

FADDEYEV, D. K. & VOROBJEV, N. N. (Leningrad) **1.1** (-.-)

Continualisation of conditional probabilities—*In Russian*
Teor. Veroyat. Primen. (1961) **6**, 116-118

The probability measure μ on finite set R is called interior if $\mu(a)>0$ for any $a \in R$. The set of all interior measures on R is denoted by $W(R)$. The following theorem is proved:

there exists a mapping ϕ of $W(R)$ into Euclidean space E of suitable dimension with two properties:

(i) All conditional probabilities

$$\mu(a \mid A) = \frac{\mu(a)}{\mu(A)} (a \in A \subset R)$$

are uniformly continuous functions of $\phi\mu$ on the whole set $\phi W(R)$ in the sense of the metric on E

(ii) The closure of $\phi W(R)$ in E is homeomorphic to the closed simplex of suitable dimension.

(D. K. Faddeyev)

2/440

HÁJEK, J. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **1.5** (4.9)

Limiting distributions in simple random sampling from a finite population—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1960) **5**, 361-374 (5 references)

The author considers an infinite sequence of simple random sampling experiments without replacement, the νth of which has the size n_ν and refers to a population of size N_ν with values $y_{\nu 1}, \ldots, y_{\nu N_\nu}$. Let ξ_ν be the sum of $y_{\nu i}$'s in the sample. Necessary and sufficient conditions concerning $\{y_\nu, n_\nu, N_\nu\}$ are given under which the distribution function of ξ_ν converges to a limiting distribution law: $n_\nu \to \infty$ and $N_\nu - n_\nu \to \infty$. This result contains as a special case that of Erdös & Rényi [*Publ. Math. Inst. Hung. Acad. Sci.* (1959) **4**, 49-61; abstracted in this journal No. **1**/171, **1.5**] for the convergence to the normal distribution law. In addition it is shown that the condition given by them is not only sufficient but also necessary.

If n_ν/N_ν does not converge to 0, normal limiting distribution and limiting distribution formed as a convolution of a normal distribution and some two-points distributions are possible. If n_ν/N_ν converges to 0, the variance preserving limiting distribution may be only infinitely divisible. The conditions for this are the same as if the sampling were carried out with replacement.

The proof is based on a model in which an experiment producing simultaneously a simple random sample and a so-called " Poisson sample " is defined. In " Poisson sampling ", each unit of the population has independently the probability n_ν/N_ν of being selected; the sample size is a random variable with expectation n_ν. The experiment is defined in the manner that the simple random sample should contain the Poisson sample or vice versa. Let $\bar{Y}_\nu = \sum_{i=1}^{N_\nu} y_{\nu i}$ and η and η^* be the sums of the differences $y_{\nu i} - \bar{Y}_\nu$ for the simple random sampling elements and for the Poisson sampling elements, respectively. Then the following inequality holds:

$$\frac{\mathscr{E}(\eta - \eta^*)^2}{\mathrm{D}\eta^*} \leqq \sqrt{\left(\frac{1}{n_\nu} + \frac{1}{N_\nu - n_\nu}\right)}.$$

As η^* can be regarded as the sum of independent terms the main theorems can be proved by application of the well-known theory of independent random variables.

(K. Sarkadi)

2/441

HÁJEK, J. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **1.5** (4.9)

Necessary and sufficient conditions for asymptotic normality of a scalar product of two randomly permutated vectors—*In English*

Second Hungarian Mathematical Congress (1961)

In this paper the author presumes a sequence of couples of vectors with real components:

$$\left\{(a_{\nu 1}, \ldots, a_{\nu N_\nu}), (b_{\nu 1}, \ldots, b_{\nu N_\nu})\right\}_{\nu=1}^{\infty}.$$

Let $(X_{\nu 1}, \ldots, X_{\nu N_\nu})$ be a random vector that takes on the $N_\nu!$ permutations of $(a_{\nu 1}, \ldots, a_{\nu N_\nu})$ with equal probabilities. Consider the statistic

$$T_\nu = \sum_{i=1}^{N_\nu} b_{\nu i} X_{\nu i}$$

and search for conditions of asymptotic ($\nu \to \infty$) normality of T_ν with respective parameters $\mathscr{E}(T_\nu)$ and $\mathrm{D}(T_\nu)$.

This problem was first formulated by Wald & Wolfowitz in 1944 who also derived a sufficient condition which was strengthened by Noether in 1949. The most far-reaching result was achieved by Motoff in 1957 who proved that the Lindeberg type is sufficient in fact, not only in the problem under consideration, but also in its generalisation by Hoeffding. The Lindeberg type condition is also necessary, however, as is shown in the following theorem:

provided that the a's and the b's satisfy a particular limiting condition, the statistic T_ν is asymptotically normal with parameters $\mathscr{E}(T_\nu)$ and $\mathrm{D}(T_\nu)$ if, and only if,

$$\lim \frac{\sum\sum_{(i,j) \in S_{\nu\tau}} (a_{\nu i} - \bar{a}_\nu)^2 (b_{\nu j} - \bar{b}_\nu)^2}{\sum_{i=1}^{N_\nu} \sum_{j=1}^{N_\nu} (a_{\nu i} - \bar{a}_\nu)^2 (b_{\nu j} - \bar{b}_\nu)^2} = 0$$

for every $\tau = 0$, where

$$S_{\nu\tau} = \left\{ (i, j) : (a_{\nu i} - \bar{a}_\nu)^2 (b_{\nu j} - \bar{b}_\nu)^2 > \tau^2 1/N_\nu \sum_{i=1}^{N_\nu} (a_{\nu i} - \bar{a}_\nu)^2 \sum_{i=1}^{N_\nu} (b_{\nu i} - \bar{b}_\nu)^2 \right\};$$

$$\bar{a}_\nu = 1/N_\nu \sum_{i=1}^{N_\nu} a_{\nu i}, \quad \bar{b}_\nu = 1/N_\nu \sum_{i=1}^{N_\nu} b_{\nu i}.$$

(J. Hájek)

2/442

HASMINSKI, R. Z. (Moscow) **1.5** (10.1)

On limit distributions for sums of conditionally independent random variables—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 119-125 (13 references)

This paper deals with limit distributions for sums η_n which become independent when a certain path $x_n(n = 0, 1, 2, \ldots)$ of a Markoff chain is defined. The dependence between $\{\eta_n\}$ and $\{X_n\}$ is expressed more exactly by

$$\Pr\{\eta_1 < z_1, \eta_2 < z_2, \ldots, \eta_n < z_n \mid X_0, X_1, \ldots, X_n\}$$
$$= \Pr\{\eta_1 < z_1 \mid X_0, X_1\}\Pr\{\eta_2 < x_2 \mid X_0, X_1, X_2\} \times \ldots \times \Pr\{\eta_n < z_n \mid X_0, X_1, \ldots, X_n\}.$$

Let X_s be the path of a continuous Markoff process. It is shown that the study of the limit distributions of $\zeta(t) = \int_0^t f(X_s)ds$ as $t \to \infty$ can be reduced to the study of limit distributions of sums η_n. This reduction is illustrated for the case where X_s is a single-dimensional diffusion process. The limit distribution for $\zeta(t)$ coincides with distributions stated by Feller [*Trans. Amer. Math. Soc.* (1949) **67**, 98-119].

(R. Z. Hasminski)

2/443

LINNIK, Y. V. (Steklov Institute of Mathematics, Leningrad) **1.5** (2.3)

Some new limit theorems for the sums of independent random variables—*In English*

Second Hungarian Mathematical Congress (1961) (2 references)

This is a somewhat extended version of the report sent to the fourth Berkeley Symposium, and briefly reported elsewhere [*Dokl. Acad. Nauk, SSSR* (1960) **133**]. The zones of integral and normal convergence for big deviations were considered and Petroff [*Dokl. Acad. Nauk, SSSR* (1960) **134**], using a suitable extension of the author's method, succeeded in obtaining " satisfactory " necessary and sufficient conditions for the convergence to the set of " Cramérian tails ". The above-mentioned results on normal convergence follow from the last results as a particular case.

The problem of finding the conditions for convergence for big deviations to the set of laws other than stable laws and " Cramérian tails " still remains open.

(Y. V. Linnik)

2/444

MOGYORÓDI, J. (Eötvös University, Budapest) **1.5** (1.6)

On limiting distributions for sums of a random number of independent random variables—*In English*

Second Hungarian Mathematical Congress (1961)

Let us consider a sequence $\{\xi_n\}$ of independent random variables and put $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$. Let $\{B_n\}$ be a sequence of positive numbers tending to infinity such that $\Pr(\zeta_n/B_n < x)$ tends to a proper distribution function $F(x)$ as $n \to +\infty$. Further, let $\{\nu_n\}$ be a sequence of positive integer-valued random variables. We mention that nothing is supposed about the dependence of ν_n on the random variables ξ_k.

The following two theorems hold:

Theorem 1. Let us suppose that ν_n/n converges in probability to one as $n \to +\infty$. In order that

$$[(\zeta_{\nu_n}/B_n) < X]$$

should converge to the law $F(x)$, it is necessary and sufficient that the following condition

$$\lim_{\delta \to 0} \limsup_{n \to \infty} B_n/B_{[n(1+\delta)]}$$

be satisfied. This theorem is in a certain sense a generalisation of a theorem by Anscombe [*Proc. Camb. Phil. Soc.* (1952) **48**, 600-607].

Theorem 2. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$, where the random variables ξ_k are independent and identically distributed with mean value zero, variance one and let $\mathscr{E}(\xi_k^4) < A$: A is independent of k. Let us suppose that ν_n/n converges in probability to a positive random variable. Then

$$\lim_{n \to +\infty} \Pr\left(\frac{\zeta_{\nu_n}}{\sqrt{\nu_n}} < x\right) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

This theorem is a generalisation of a theorem by Rényi [*Acta Math. Acad. Sci. Hung.* (1960) **11**, 97-102; abstracted in this journal, No. **1**/557, **1.5**].

(J. Mogyoródi)

2/445

NAGAEV, S. V. (Tashkent) **1.6** (1.5)

More exact statements of limit theorems for homogeneous Markov chains—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 67-86 (11 references)

This paper contains several theorems defining more precisely the convergence of homogeneous Markoff chains with an abstract state space to the Gaussian distribution. Moreover, limit theorems for large deviations are proved. The proofs are analogous to the ones in an article of Nagaev [*Teor. Veroyat. Primen.* (1957) **2**, 389-416].

(S. V. Nagaev)

2/446

PALÁSTI, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.4** (1.5)

On some random space filling problems—*In English*

Second Hungarian Mathematical Congress (1961)

Place as many as possible disjoint domains at random, and with uniform distribution, into a rectangle with sides x and y: these disjoint domains being congruent and parallel to a given convex domain D. The question arises, how a large part of the rectangle will be filled on average with these domains?

Let us denote by $M(x, y)$ the mean value of the number of domains placed in the above-mentioned way. The plausible hypothesis is set up that there exists a constant $B>0$ such that

$$| M(x_1+x_2, y)-M(x_1, y)-M(x_2, y) | \leqq By$$

and $$| M(x, y_1+y_2)-M(x, y_1)-M(x, y_2) | \leqq Bx.$$

A theorem of Hyers concerning functional inequalities is generalised for two dimensions as follows: let $f(x, y)$ be a Borel-measurable function of two variables, satisfying the following conditions:

$|f(x_1+x_2, y)-f(x_1, y)-f(x_2, y)|\leqq By$ for $x_1\geqq 0$, $x_2\geqq 0$, $y\geqq 0$ and $| f(x, y_1+y_2)-f(x, y_1)-f(x, y_2) | \leqq Bx$ for $x\geqq 0$, $y_1\geqq 0$, $y_2\geqq 0$.

$B>0$ is a constant, not depending on the variables x_i, y_i, $(i = 1, 2)$ and there exists a constant $\rho>0$ such that $f(x, y) = 0$ if $0<x<\rho$ or $0<y<\rho$ then

$$| f(x, y)-\alpha xy | \leqq B(x+y)$$

where α is constant. Thus

$$\lim_{x,\, y\to\infty} f(x, y)/xy = \alpha$$

with the aid of which the limit of the above hypothesis is shown to be

$$\lim_{x,\, y\to\infty} M(x, y)/xy = \alpha(D)$$

If D equals the unit square S_1, then $\alpha(S_1) = c^2$ where the constant c is the constant obtained in the analogous problem for one dimension which has been solved by Rényi [*Mag. Tud. Akad. Kut. Mat. Intézet. Közlemenyei* (1958) 109-127: abstracted in this journal No. **1**/18, **1.4**],

$$c = 2\int_0^\infty (1-e^{-t}) \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\, du\right) dt \sim 0{\cdot}748$$

and thus the value of c^2 is approximately equal to 0·56. This means that about 56 per cent. of the area of a large rectangle will be covered by randomly placed unit squares, with sides parallel to those of the rectangle.

Analogously, it is conjectured that, for the n-dimensional problem, on average the c^nth per cent. of the volume of a large n-dimensional parallelotope will be covered by n-dimensional unit cubes, with sides parallel to those of the parallelotope, and placed at random.

(I. Palásti)

2/447

PREDETTI, A. (Catholic University of Milan) **1.2** (1.1)

Characteristic features of a general scheme of repeated trials with dependent probabilities—*In Italian*

Riv. Ital. Econ. Demogr. Statist. (1960) **14**, 85-93 (2 references)

The author considers a particular scheme by Woodbury according to which the law of dependency of the probabilities is linear.

After having proved that this scheme can be considered as resulting from the application of a Pólya scheme to a Poisson scheme, he determines the mean and variance of the number of successes. In particular, it is shown that the said mean value has an expression formally analogous to the compounded amount, principal value and interest, of a rent.

The author then gives a procedure by which the parameters included in the formulae obtaining for the mean and variance can be determined approximately.

(V. Levis)

2/448

PROHOROFF, Y. V. & SAZONOFF, V. V. (Moscow) **1.1** (–.–)

Some results associated with Bochner's theorem—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 87-93 (9 references)

In this paper the authors discuss some results associated with Bochner's theorem. Let $\{P_\alpha\}$ be a family of probability distributions in a separable Hilbert space; or more generally, in a space $X = Y^*$ conjugate to countably-Hilbert space Y: let $\{\chi_\alpha\}$ be the family of corresponding characteristic functionals. They investigate whether or not there exists a locally convex topology T having the property that the relative compactness of $\{P_\alpha\}$ is equivalent to uniform, with respect to α, continuity of $\{\chi_\alpha\}$.

They prove that a topology of this kind does not exist except for the case of countably-Hilbert nuclear space Y.

(Y. V. Prohoroff)

2/449

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.5** (1.3)

Limit theorems concerning random walk problems—*In Hungarian*

Magy. Tud. Akad. III Oszt. Közl. (1960) **10**, 149-169 (20 references, 1 figure)

In this paper the author summarises the most important results in connection with the random walk; first of all on the real line. He gives new proofs of some known results.

For example, he applies the generalised Borel-Cantelli lemma [Erdös & Rényi, " On Cantor's series with convergent $\Sigma 1/q_n$ ", *Ann. Univ. Sci. Budapest*, Sect. Math. (1959) **2**, 93-100; abstracted in this journal No. **1**/543, **1.5**] to prove the recurrence theorem of Pólya. A simple proof of the *arc sine* law is obtained by showing that the characteristic function of the number of steps among the first n steps, after which the random walk point is to the right of the origin, is essentially equal to a Legendre polynomial.

(P. Révész)

2/450

REIMANN, J. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.3** (0.6)

Matrix-theoretical treatment of random walk problems—*In English*

Second Hungarian Mathematical Congress (1961)

Random walk on a lattice-point region of an n-dimensional-space with absorbing and reflecting barriers, leads to hyper-matrices, which can be expressed by the direct products of the matrices obtained in the one-dimensional case. Thus, the results obtained for one dimension can be applied in the investigations of random walks of higher dimensions.

With the matrix method outlined above, the following problems are solved:

(i) in the case of random walk in a plane and in space, respectively, on a lattice-rectangle with absorbing barriers, the probability is calculated that a particle starting from a given interior lattice-point will arrive in steps to a predescribed interior or absorbing lattice-point,

(ii) in the case of absorbing barriers on a n-dimensional rectangle with given size and given starting point, a formula is given for the probability of the particle being " alive " after N steps,

(iii) in the case of symmetrical random walk, the expected value of the number of steps before absorption is calculated,

(iv) for an asymmetrical random walk in two and three dimensions with absorbing barriers, the probability that a particle starting in an interior point will be absorbed after an undetermined number of steps in a predescribed absorbing point is calculated. That is, in case of the random walk in a plane or on a lattice rectangle consisting of $m+2$ rows and $n+2$ columns, this probability is

$$U_{(i,\, j)\to(\mu,\, 0)} = \frac{2}{m+1}\left(\frac{q}{p}\right)^{\frac{j-1}{2}}\left(\frac{r}{s}\right)^{\frac{i-\mu}{2}}\frac{q}{\sqrt{(rs)}}\sum_{l=1}^{m}\sin\frac{il\pi}{m+1}\sin\frac{\mu l\pi}{m+1}\,\frac{sh(n+1-j)\vartheta_l}{sh(n+1)\vartheta_l};$$

where

$$2ch\vartheta_l = \frac{1}{\sqrt{(rs)}} - 2\sqrt{\left(\frac{pq}{rs}\right)}\cos\frac{k\pi}{n+1}.$$

This probability has also been determined for the symmetrical random walk by McCrea and Whipple.

(v) The problems mentioned before are also answered in case of partly absorbing and partly reflecting barriers.

continued

REIMANN, J. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.3** (0.6)

Matrix-theoretical treatment of random walk problems—*In English* *continued*

Second Hungarian Mathematical Congress (1961)

(vi) An iterative matrix-theoretical method is given for the solution of the problem mentioned in (iv) above, in the case of a lattice-point region of arbitrary form.

(vii) In case of random walk on a lattice-rectangle with reflecting barriers, explicit formulae are given for the solution of the problem stated in (i).

(viii) A special case of random walk in a plane is investigated; namely, when the probability of moving in different directions is not constant, as in above cases, but is a function of the coordinates. In connection with this problem, the Ehrenfest-model of diffusion is generalised.

(J. Reimann)

RÉVÉSZ, P. (Eötvös University, Budapest) **1.6** (10.1)

A remark on the random ergodic theorem—*In English*

Second Hungarian Mathematical Congress (1961)

In this paper the author poses a problem in connection with a theorem stated by Ulam & Neumann and later discussed by Rényi; the original theorem is as follows:

let $\mathscr{T}$ be a measure space of the measure preserving transformations defined on $\{X, \mathscr{S}, \mu\}$ with a probability measure P. Let us consider the product space

$$\mathscr{T}^* = \mathscr{T}_1 \times \mathscr{T}_2 \times \ldots,$$

where $\mathscr{T}_i = \mathscr{T} (i = 1, 2, \ldots)$ and the product measure $P^* = P_1 \times P_2 \times \ldots$.

Let $f(x) \in L^1_{(X, \mu)}$ then

$$\Pr\left\{1/n \sum_{k=1}^{n} f(T_k \ldots T_1 x) \to f^*(x) \text{ almost everywhere}\right\} = 1$$

where $(T_1, T_2, \ldots)$ is a point of $\mathscr{T}^*$ and $f^*(x) \in L^2_{(X, \mu)}$.

Rényi raised another problem: if there is a sequence $T_1, T_2, \ldots$ of measure preserving transformations, under what conditions does $1/n \sum_{k=1}^{n} f(T_k \ldots T_1 x)$ converge almost everywhere or in mean? An interesting problem in connection with this question is the following: if from the set $\mathscr{T}$ a sequence $T_1, T_2, \ldots$ is chosen at random, not necessarily with the same distributions; for instance with the distributions $P_1, P_2, \ldots$, but independently, under what conditions does $1/n \sum_{k=1}^{n} f(T_k \ldots T_1 x) \to f^*(x)$ hold almost everywhere or in mean with probability one? Some conditions are given: for instance, it is proved that if

$$\left\| \int_{T \in \mathscr{T}} f(T_x) dP_i \right\|_{L^2_{(X, \mu)}} \leqq m_i \| f \|_{L^2_{(X, \mu)}} \qquad m_i \leqq 1 - 1/i^{1-\varepsilon}$$

$$0 < \varepsilon < 1$$

for every $f(x)$ for which $f \in L^2$, $\int f = 0$ then the result holds. It is used to prove a strong law of large numbers in the theory of Markoff chains.

(P. Révész)

2/453

SARKADI, K. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.3** (5.6)

On random walk problems—*In English*

Second Hungarian Mathematical Congress (1961) (2 references)

In this paper the author gives a new proof for a known theorem concerning the so-called Galton-test [see Feller, *An introduction to probability theory and its applications* (1957) New York: Wiley, 68-73] and some connected problems are considered. For the above-mentioned theorem a combinatorial proof [Hodges, " Galton's rank-order test " *Biometrika* (1955) **42**, 261-262] is already known; the proof given here is simpler.

The combinatorial formulation of the mentioned theorems are as follows:

let us consider the random sequence S of n $(+1)$'s and n (-1)'s. Let s_k be the sum of the first k members of the sequence and let us denote by $v = v(S)$ the smallest, and by $w = w(S)$ a randomly chosen number from the values for which the relationship

$$s_v = s_w = \max_k s_k$$

holds. Further, let $u = u(S)$ be the number of positive numbers in the sequence $s_1, s_3, \ldots, s_{2n-1}$.

The theorem concerning Galton's test states that

$$\Pr(u = r) = 1/(r+1) \quad (r = 0, 1, \ldots, n).$$

A theorem of Sparre Andersen states that

$$\Pr(v = 2r) = \Pr(v = 2r+1) \quad (r = 1, 2, \ldots, n-1).$$

The following theorem is also proved:

$$\Pr(w = r) = 1/2n.$$

The above theorems are proved by giving one to one transformations mapping the sequences S into sequences S' and S'' for which $u(S') = u(S) - 1$, $v(S'') = v(S) - 1$ and $w(S'') = w(S) - 1$ respectively.

The third theorem can be extended for a sequence of n $(1/n)$'s and m $(1/m)$'s as well. In statistical terminology: the theorem can be extended for samples of unequal size.

(K. Sarkadi)

2/454

TINTNER, G. (Iowa State University, Ames) **1.8** (0.7)

The application of decision theory of probability to a simple inventory problem—*In English*

Trab. Estadíst. (1959) **10**, 239-247 (12 references)

The author uses an elementary form of the inventory problem in order to illustrate some ideas in the field of decision theory and probability.

In order to illustrate this he supposes a commodity that is bought for $\$b_2$ and sold for $\$b_1$ per indivisible unit. Assuming some hypotheses about sales it is necessary to determine the optimum number of items to be bought. Three cases are discussed in this paper:

(i) A probability distribution of sales exists and is known. The objective might be either the maximisation of the mathematical expectation of profit, or the minimisation of variance with equal mathematical expectation.

(ii) The probability distribution is not known. The problem might be considered as a game between the individual and nature. The objective might be reached with either the minimax strategy or the minimum regret; regret equivalent to loss which is represented by the difference between the profits actually made and the maximum profit which might have been realised by the given strategy. Alternatively one might maximise a weighted average of the maximum and minimum profit which can be realised.

(iii) The probability distribution of the state of nature is not known and the optimum is obtained by use of an *a priori* probability distribution: either an equal probability is assumed for possible demand values, or an application of Carnap's theory of probability is utilised for hypotheses about probability of possible demand values.

(J. M. Garcia)

2/455

VIETORIS, L. (Math. Inst., Universität Innsbruck) **1.0** (4.0)

Remarks and estimates concerning induction—*In German*

Monatsh. Math. (1960) **64**, 233-250 (14 references)

Technically, this is in essence a sharpening along lines suggested by J. V. Uspensky [*Introduction to Mathematical Probability* (1937) New York: McGraw-Hill], of Bernoulli's (*Ars conjectandi*) estimation of probabilities through observed relative frequencies.

More informally, the topic is discussed in context with a degree-of-confirmation concept of probability; no mention is, however, made of the standard literature, e.g., Carnap [*The Continuum of Inductive Methods* (1952) Chicago: University of Chicago Press]. While the author explains a degree of confirmation as " a relation between the accredited hypotheses and the accredited affirmation " and is therefore in agreement with Carnap's theory of *c*-functions, it appears plausible that quite different points of departure, like that of de Finetti or L. J. Savage would do equally well for Vietoris' specific purpose.

In particular, the author takes issue with Bertrand Russell's position [*Das menschliche Wissen*, Zurich: Holle] regarding the justification of inductive inferences in principle. According to Russell some extra-logical properties of the real world alone, not mathematical probability theory, must provide that justification. It is the author's contention that extra-logical properties, called " primary " by him do indeed provide that justification, but via the relative frequencies which depend on " physical constants " and which, as " degrees of possibility," form the conventional basis of probability theory and statistics.

Of more technical content is the author's rejection of a probabilistic analogue of statement-logical contraposition (i.e., " if *A*, then *B*; then if non-*B*, non-*A* "), by means of an inverted use of the Clopper-Pearson formula [*Biometrika* (1934) **26**, 404] for confidence limits for which tables have been provided by Dudin-Barkovski & N. V. Smirnoff [*Teoria veroyatnostei i matematicheskaya statistika v tekhnike* (1955) Moscow: Gittl.

(E. M. Fels)

2/456

VOGEL, W. (Univ. Tübingen, Germany and Univ. Chicago) **1.8** (9.7)

A sequential design for the Two-Armed Bandit—*In English*

Ann. Math. Statist. (1960) **31**, 430-443 (8 references)

Consider two independent Bernoulli variates X and Y with unknown means p and q respectively. The problem is to maximise the expected sum of n independent observations, each being from X or Y, where after each observation the experimenter may select the variate to be sampled. The author restricts himself to the following situation: observe alternate pairs, a decision being made at each stage by watching the random walk $U_k = X_1 - Y_1 + \ldots + X_k - Y_k$ with absorbing barriers at $\pm\alpha$. If $-\alpha < U_k < \alpha$ obtains, observe another pair: if $U_k \geqq \alpha (\leqq -\alpha)$ obtains, stop observing pairs and observe only X's (Y's) for the remainder of the n steps. Let K be the step at which either absorption occurs or else $K = \mu$ $(n = 2\mu)$.

The best possible outcome in expectation is $n\sigma$ where $\sigma = \max(p, q)$. The loss function is defined as $L_n = n\sigma - W$, where W is the expected sum of all n steps. It is proved that $L_n = (\sigma - \tau)[n + (u^\alpha - 1)\mathcal{E}K]/(u^\alpha + 1)$, where $\tau = p + q - \sigma$ and $u = \sigma(1-\tau)/\tau(1-\sigma)$. A strategy consists simply of a choice of a positive integer $\alpha \leqq \mu$. A strategy minimising L_n cannot be found without some *a priori* knowledge of p and q. This is the approach of Bradt, Johnson & Karlin [*Ann. Math. Statist.* (1956) **27**, 1060-1074].

Using $\mathcal{E}U_K = (p-q)\mathcal{E}K$, a slight generalisation is proved in this paper and another formula, derived in terms of absorbing probabilities at each barrier and the probability of stopping at μ, the author obtains the inequality $\mathcal{E}K \leqq \alpha(u^\alpha - 1)/(\sigma - \tau)(u^\alpha + 1)$ with the left-hand side approaching the right as $n \to \infty$. Let M_n be the approximate loss-function obtained by substituting the upper bound for $\mathcal{E}K$ into the expression for L_n. The author then uses M_n to choose an optimum strategy in the minimax sense.

Now M_n is a function of α, σ, τ and by regarding α as a continuous variable on $0 < \alpha \leqq \mu$, the author obtains the theorem for $n \geqq 4$ that the minimum over α of the maximum over (σ, τ) of M_n equals the quantity attained by interchanging these extrema. Calling the saddle point coordinates $\alpha(n)$, $\sigma(n)$, $\tau(n)$ the author gives their

continued

VOGEL, W. (Univ. Tübingen, Germany and Univ. Chicago) **1.8** (9.7)

A sequential design for the Two-Armed Bandit—*In English*

Ann. Math. Statist. (1960) **31**, 430-443 (8 references)

continued

asymptotic behaviour as $n \to \infty$ as well as $\sigma(n) - \tau(n)$, and $\alpha(n) \log u(n)$. This allows the author to propose the simple formula $\alpha(n) = (0{\cdot}292\sqrt{n})$ for $n > 100$ as the optimum minimax strategy. Some considerations are given as to when the approximation M_n is good and the result applicable, but no error estimates are given.

Another approximation for L_n, for n large, is obtained heuristically by considering a sequence of random walks of the type discussed, approaching a Weiner-Process, and by choosing the appropriate absorbing boundaries and using well-known formulae for the expected time to reach these boundaries. On this basis the author conjectures a limiting formula for $L_n/\sqrt{n}$ with α_n, $\sigma_n - \tau_n$ being $O(\sqrt{n})$ and $\sigma_n, \tau_n \to \rho \in (0, 1)$. The author then adduces some evidence for his suspicion that this limiting formula has a saddle point. In the last section some conditions are discussed which allow this procedure to be generalised to other than Bernoulli variates.

(S. C. Saunders)

AMATO, V. (Faculty of Economics, University of Catania) **2.5** (0.2)

A new theoretical scheme of a factorial type and the distribution of plural births—*In Italian*

Riv. Ital. Econ. Demogr. Statist. (1960) **14**, 33-64 (9 references, 14 tables)

The author, noting the systematic difference between the empirical frequencies of some distributions of plural births and the theoretical frequencies derived from the geometric law, suggests a less simple model which appears to give a better " fit " to the actual distributions.

Such a model belongs to the class of factorial models with dependent probabilities, and its choice is justified by the ample heterogeneity of the partial distributions entering into the total distribution of the observed phenomenon and by the dependency between the relevant events.

The author first shows the better " fit " obtaining with the theoretical distribution he suggests and then proves how this distribution converges to the geometric one. A simple procedure is also given for calculating the moments.

(V. Levis)

2/459

BARDWELL, G. E. (University of Denver, Colorado) **2.1** (2.4)

On certain characteristics of discrete distributions—*In English*

Biometrika (1960) **47**, 473-476 (1 reference)

The author starts from the remark, " It has been stated that for the hypergeometric, binomial, Poisson and geometric distributions the relationship holds that the mean deviation is equal to twice the variance multiplied by the frequency function at $[\mu]$, where $[\mu]$ is the greatest integer not greater than the mean ".

He is concerned to point out that this relationship is true for the binomial and Poisson, true for the geometric distribution for integral values of μ, but does not hold for the hypergeometric distributions. Further properties of the discontinuous frequency functions mentioned above are derived.

(Florence N. David)

2/460

BOROVKOFF, A. A. **2.1** (-.-)

Local theorems and moments for maximums of sums of lattice restricted components—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 108-110 (1 reference)

Let $\xi_1, \xi_2, \ldots,$ be independent latticed random variables;

$$|\xi_k| < M, \; \bar{s}_n = \max_{1 \leq \nu \leq n} (0; \; \xi_1 + \ldots + \xi_\nu).$$

Both the formulae for $\Pr(\bar{s}_n = x)$ and for the first moments of s_n are obtained by the author in this paper.

(A. A. Borovkoff)

2/461

BUEHLER, R. J. (Statist. Lab., Iowa State Univ., Ames, Iowa) **2.2** (6.8)

An application of regression to frequency graduation—*In English*

Biometrics (1960) **16**, 659-670 (11 references, 3 tables)

Given observed values (x) grouped in a frequency distribution, it is desired to derive a function $\phi(x)$ which gives a smooth representation in terms of some standard distribution $g(x)$. Hammersley & Morton [*Biometrika* (1954) **41**, 296-301] utilised a method of weighted least-squares to determine the parameters α and β in the transformation $\phi(x) = \alpha + \beta x$, where $g(x)$ is arbitrary.

In this paper the author generalises this transformation to

$$\phi(x) = \sum_{i=1}^{m} \beta_i \phi_i(x),$$

where the ϕ_i are any number m of arbitrary specified functions. The β_i are asymptotically efficient estimates obtained by the method of weighted least-squares proposed by Aitken [*Proc. Roy. Soc. Edinb.* (1935) **55**, 42-48]. Simultaneous asymptotic minimisation of chi-square is also noted. A major advantage of the method is that estimates of the percentile points y_i of $g(x)$ are given by an explicit formula rather than the implicit formula used by Edgeworth [*J. R. Statist. Soc.* (1898) **61**, 671-699].

The author proposes an F test which is helpful in determining the correct number of parameters in the transformation. However, the use of a multi-parameter expression of the form shown above has the inherent difficulty that ϕ may have maxima or minima which leads to a multi-valued inverse function $1/\phi$. The problem is somewhat resolved if it can be shown that ϕ is monotone in the range of interest.

As an example, the method was applied to the breadth-of-bean data of Johannsen reported for example by M. G. Kendall & Stuart [*The Advanced Theory of Statistics* (1958) London: Griffin] to effect a cubic transformation ($m = 4$) to normality. Details of the calculations are shown. Graduated percentile points obtained by this method are compared with fitting by three other types of four-parameter equations. Both the Pearson type IV curve obtained by Pretorius [*Biometrika* (1930) **22**, 109-223] and that obtained by Johnson [*Biometrika* (1949) **36**, 149-176] using a type S_U translation give superior fits using χ^2 as a rough guide. For the purpose of transforming to normality the method proposed in this paper is superior to that of unweighted least-squares.

(J. E. Dunn)

2/462

COHEN, A. C. (University of Georgia, Athens, U.S.A.) **2.8** (4.3)

An extension of a truncated Poisson distribution—*In English*

Biometrics (1960) **16**, 446-450 (6 references, 1 table)

This paper is concerned with the estimation of parameters for a truncated Poisson distribution with missing zero class. Although the distribution to be considered might be studied from a strictly abstract point of view, the author points out that it is desirable to keep in mind the practical example of the distribution of organisms among colony sites under circumstances such that no migration occurs between colonies.

The maximum-likelihood estimate is obtained for one parameter. The maximum likelihood estimating equation is obtained and easily solved for the second parameter by interpolating linearly in Table I of an earlier article by Cohen [*Biometrics* (1960) **16**, 203-211; abstracted in this journal No. **2**/46, **4.3**].

The asymptotic variance-covariance matrix of the two parameters is obtained and the parameters are shown to be asymptotically independent.

The results of this paper are illustrated by using an example from Beall & Rescia [*Biometrics* (1953) **9**, 354-386]. A tabulation of the observed data is presented together with expected frequencies using results of this paper, and for comparison expected frequencies based on a complete Poisson distribution and on five contagious distributions considered by Beall & Rescia. As modified here, the superiority of the truncated Poisson distribution is clearly demonstrated in fitting the observed distribution of this example.

(L. J. Walker)

2/463

CRAWFORD, J. R. & WALSH, J. E. (System Development Corp., Cal.) **2.7** (2.1)

Empirical examination of Edgeworth series—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 13-26 (4 references, 3 tables)

In this paper the authors describe a numerical investigation carried out for five types of probability approximations by using the first seven terms of the Edgeworth series expansion for the distribution of a continuous random variable T. These approximations are applied to three types of probability expressions $F(t) = \Pr(T \leqq t)$, $F(t)-F(-t) = \Pr(-t \leqq T \leqq t)$, and $F(t)-F(-t+1) = \Pr(-t+1 \leqq T < t)$, where T is taken to have zero mean and unit variance.

Calculations are performed for each combination of certain values of the third, fourth, and the fifth moment about the mean. For those t values in the set $-4{\cdot}00$ $(0{\cdot}25)$ $4{\cdot}00$, the authors obtain significant limits t_1, t_2 $(t_1 < t_2)$ which indicate that a probability expression satisfies the required monotonicity conditions and neither exceeds unity nor falls below zero for computed t such that $t_1 \leqq t \leqq t_2$. In addition to the significant limits, the values of $F(0)$ and $F(1{\cdot}75)-F(-1{\cdot}75)$ are given for each case investigated: the tables of these results are also given.

(M. Huzii)

2/464

GAEDE, K. W. (Technische Hochschule, München) **2.0** (2.2)

On the distribution of the roots of random algebraic equations—*In German*

Math. Nachr. (1960) **21**, 81-107 (6 references, 10 figures)

In this paper the author discusses a polynomial of degree n in one variable x with the leading coefficient 1: the other coefficients $A_1, A_2, \ldots, A_n$ are of the form $A_k = S_k + iT_k$, the S_k and T_k being real random variables. It is supposed that the joint distribution function of these $2n$ random variables is known and that the density function exists. The roots are suitably arranged in order to get a one-to-one correspondence between the coefficients and the roots of the polynomial. The functional determinant of the related transformation can be explicitly computed. With this, the joint frequency function of the real and imaginary parts of the roots can be calculated.

In the case where all random coefficients are always real, they cannot be submitted to the above-mentioned research because of the supposed existence of the density function. It is assumed that the joint density function of the n real random variables $S_1, \ldots, S_n$ exists and is known. The frequency function of the roots can also be computed; but some cases have to be distinguished according to the number of pairs of conjugate complex roots. An additional and necessary condition that all roots are real for all realisations of the coefficients is stated.

For the treatment of round-off errors of the coefficients it is possible to assume that the density function of the coefficients is constant in a certain bounded domain and is otherwise equal to zero. The density function of the roots does not then assume its maximum in the interior of the domain in which the density function is not identically equal to zero.

The second part of the present paper is exclusively devoted to the cubic equation with real random coefficients and more especially to the cubic equation in normal form; without a quadratic term. The investigations mentioned above are carried out in full detail: examples are given assuming the distribution of the coefficients to be normal, rectangular and negative exponential.

(W. Uhlmann)

2/465

GRIMM, H. (Inst. für Mikrobiologie und exp Therapie, Jena) **2.2** (6.8)

Transformation of variates—*In German*

Biom. Zeit. (1960) **2**, 164-182 (159 references, 3 figures)

This paper is concerned with the transformation of discrete random variates for use in analysis of variance, regression analysis and other parametric methods based on normal distribution theory.

In the introductory paragraphs the assumptions underlying parametric methods are discussed and tests are described for the validity of these assumptions. An extensive catalogue of usual transformations is given along with methods to show when and how to use them. The substantial bibliography is arranged by subjects and author's names.

(R. Wette)

2/466

GUMBEL, E. J. (Columbia University, New York) **2.5** (2.1)

Bivariate exponential distributions—*In English*

J. Amer. Statist. Ass. (1960) **55**, 698-707 (7 references, 3 figures)

A bivariate distribution is not determined by its margins. Fréchet in fact has shown that for any given margins there exist infinitely many bivariate distributions. The author defines three bivariate distributions having exponential margins. Their cumulative distribution functions are, for $x \geqq 0$, $y \geqq 0$,

$$F_1(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\delta xy} \qquad 0 \leqq \delta \leqq 1$$

$$F_2(x, y) = (1-e^{-x})(1-e^{-y})(1-\alpha e^{-x-y}) \qquad -1 \leqq \alpha \leqq 1$$

$$F_3(x, y) = 1 - e^{-x} - e^{-y} + \exp\{-(x^m + y^m)^{1/m}\} \qquad m \geqq 1$$

The first two are investigated in detail. None of the well-known properties of the bivariate normal distribution—such as ellipticity of contours, linearity of regression, or homoscedasticity are found to hold. Thus, one is cautioned against supposing that the normal case is necessarily typical.

For F_1 and F_2 the author obtains the marginal and conditional densities, conditional means and variances, the correlation coefficient, and the correlation ratio. The conditional means and standard deviations are plotted for a few special parameter values. For F_1 the conditional mean of either variable decreases to zero as the other variable increases to infinity; for F_2 it may increase or decrease depending on the sign of the correlation, which in turn depends upon the sign of α. The correlation coefficient lies in the range $-0{\cdot}40 \leqq \rho \leqq 0$ for F_1, and $-0{\cdot}25 \leqq \rho \leqq 0{\cdot}25$ for F_2. Note: there is a typographical slip in (3.6), which should read: $\mathscr{E}(x \mid y) = 1 + \frac{1}{2}\alpha - \alpha e^{-y}$.

(R. J. Buehler)

2/467

HIGUTI, Z. (Institute of Statistical Mathematics, Tokyo) **2.9** (-.-)

Remark on the weak distribution in a locally convex space. I.—*In French*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 63-67

In this paper the author gives a necessary and sufficient condition for a weak distribution of X to be realisable in a space X, if X is a countably normed space which is complete and perfect. This condition is analogous to the one in the case of the Banach space given by Getoor [" On characteristic functions of Banach space valued random variables ", *Pacific J. Math.* (1957) **7**, 885-896]. An application to the study of the random Schwartz distribution is discussed.

(M. Huzii)

2/468

ISHII, G. & HAYAKAWA, R. (Atomic Bomb Casualty Comm., Hiroshima) **2.4** (2.9)

On the compound binomial distribution—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 69-80 (8 references, 4 tables, 1 figure)

This paper deals with compound binomial and multinomial distributions and some of their applications. Let a random variable X have the positive binomial distribution $\binom{n}{x} p^x(1-p)^{n-x}$ with parameter p and suppose p varies with probability density

$$f(p)=\frac{1}{\mathrm{B}(a,\,b)}\,p^{a-1}(1-p)^{b-1},\ 0\leqq p\leqq 1$$

(Beta distribution). The overall distribution is called the binomial-beta distribution. Further, the bivariate binomial beta distribution is defined as follows:

$$\Pr(x,\,y)$$
$$=\int_0^1\binom{n_1}{x}p^x(1-p)^{n_1-1}\binom{n_2}{y}p^y(1-p)^{n_2-y}f(p)dp,$$

where $f(p)$ is the density function of a Beta distribution.

In this bivariate case, the authors obtain the conditional distribution when one of variates is fixed; also the distribution of the sum of variates. They show particularly that the regression of Y on X is linear. In the same way, they discuss the k-variate binomial beta distribution and the bivariate multinomial-beta distribution.

As applications they refer to comparisons of sex ratios in human families, analyses of absenteeism and the scoring of a game.

(Y. Suzuki)

2/469

LANCASTER, H. O. (University of Sydney) **2.3** (2.1)

The characterisation of the normal distribution—*In English*

J. Aust. Math. Soc. (1960) **1**, 368-383 (18 references)

Two lemmas are proved: (i) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors whose elements are sets of m and n mutually independent random variables related by $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$. Then, if each column of $\boldsymbol{A}$ contains two non-zero elements and if $2\leqq m\leqq n$, moments of all orders are finite. If the variables x_i are identically distributed, the condition may be weakened so that one column of $\boldsymbol{A}$ contains two non-zero terms.

(ii) If $x_1, x_2, \ldots, x_n$ are identically and independently distributed and $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is independent of $\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-negative definite real symmetric matrices and $a_{jj}b_{jj}$ is not zero for at least one j, then all moments of x_i are finite.

These lemmas are then used to prove the well-known characterisations of the normal by Herschel and by Geary and the more recent one by Skitovitch, with the aid of cumulant theory. It is shown that Herschel's (spherical symmetry) characterisation requires neither the Cramér nor the Marcinkiewicz theorem. The lemmas can also be used to complete other proofs which assume the existence, for example, of finite second moments. It is then proved that, in the class of symmetrical distributions, the Cochran or the Craig conditions for independence characterise the normal distribution; the method of proof is to show that a recursive relation holds between the cumulants of even order and that this relation must be the same as holds for the normal distribution.

(H. O. Lancaster)

MEDGYESSY, P. (Math. Inst., Hungarian Academy of Sciences, Budapest) **2.8** (2.5)

On some unsolved problems in the theory of the decomposition of superpositions of distribution functions—*In English*

Second Hungarian Mathematical Congress (1961) (1 reference)

We are given the superposition of continuous distribution functions

$$G(x) = \sum_{k=1}^{N} p_k F(x-\alpha_k;\ \beta_k)$$

$$(0<\beta_1<\ldots<\beta_N;\ \alpha_i \neq \alpha_k (i \neq k);\ i, k = 1, \ldots, N)$$

where a section of the graph of $G(x)$ and the type of $F(x-\alpha_k;\ \beta_k)$ are known; $p_k>0$, α_k, β_k are, however, unknown.

The problem of determining these unknown parameters, based on the graph section of $G(x)$, arises; it is called the "decomposition" of $G(x)$. In order to solve, at least approximately, this problem and also more complicated ones, the author has given several methods, both in papers, and in his book ["*On the decomposition of superpositions of distribution functions*" (1961) Budapest: Akadémiai Kiadó].

Some of these methods are based on the finding of a special parameter of distribution functions, the so-called "formant", whose definition is as follows: if $\phi(x;\ c)$ is a distribution function depending on the parameter c, then c is called a "formant" (a, m, c_0)

2/471

if, for $c_0 \leqq c < a$, or $a < c \leqq c_0$, c_0 is given

$$\lim_{c\to a} \phi(x;\ c) = U(x;\ m) = \begin{cases} 1 & x>m \\ & \text{if} \\ 0 & x \leqq m \end{cases}$$

a, m are constants, and $\phi(x;\ c)$ is non-degenerate if $c_0 \leqq c < a$ or if $a < c \leqq c_0$.

There are, however, types of distribution functions $F(x-\alpha_k;\ \beta_k)$ such that for a superposition formed with them, all the mentioned methods fail. An interesting type is the following:

$$F(x-\alpha_k;\ \beta_k) = \int_{-\infty}^{x-\alpha_k} \frac{\beta_k}{\pi}\left(\frac{\sin \beta_k y}{\beta_k y}\right)^2 dy.$$

It is known that the relevant density functions occur in physical optics yielding the intensity "distribution" in a screen of the light radiating from a lighted gap which is parallel to the screen. This superposition could not, however, be decomposed by any of the mentioned methods.

Another superposition which is so far undecomposable is that of Γ distribution functions of order n where

$$F(x-\alpha_k;\ \beta_k) = \int_0^{x-\alpha_k} \frac{y^{n-1} e^{-\frac{y}{\beta_k}}}{\beta_k^n \Gamma(n)}\, dy$$

and $\alpha_k \neq 0$ $(k = 1, \ldots, N)$. (P. Medgyessy)

RAMASUBBAN, T. A. (London School of Economics) **2.1** (2.4)

Some distributions arising in the study of generalised mean differences—*In English*

Biometrika (1960) **47**, 469-473 (4 references)

The author has previously defined the coefficients of mean differences of the rth order as the rth power of the modulus of the difference weighted by the probability distribution functions of the two variable and integrated (or summed when they are discrete) over the whole permissible ranges. He has previously discussed the first order (Δ_1) for the positive and negative binomial, Poisson and geometric distributions: [*Biometrika* (1958) **45**, 549-556; abstracted in this journal No. **1**/31, **2.1** and *Biometrika* (1959) **46**, 223-229; abstracted in this journal No. **1**/195, **2.1**]. He has also expressed Δ_r in reasonably compact form in a previous paper.

In this present paper he examines from a different point of view the general problem of expressing Δ_r in explicit form. Three distributions, the binomial, Poisson and the geometric are treated.

(Florence N. David)

2/472

ROGOZIN, B. A. (Moscow) **2.1** (–.–)

On the increase of dispersion of sums of independent random variables—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 106-108 (6 references)

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables,

$$Q_k(l) = \sup_x \Pr\{x \leqq \zeta_k \leqq x+l\}$$

$$Q(L) = \sup_x \Pr\{x \leqq \xi_1 + \ldots + \xi_n \leqq x+L\},$$

$$s = \sum_{k=1}^{n} [1 - Q_k(l_k)] l_k^2.$$

In this paper it is proved by the author that, where C is an absolute constant, and if $L >_{\max} l_2$, then

$$Q(L) \leqq \frac{CL}{\sqrt{s}}.$$

(B. A. Rogozin)

2/473

ROGOZIN, B. A. (Moscow) **2.1** (–.–)

On evaluating the concentration function—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 103-105 (3 references)

Let $\xi_1, \ldots, \xi_n$ be independent random variables,

$$Q_k\{l\} = \sup_x \Pr\{x \leqq \zeta_k \leqq x+l\},$$

$$Q\{L\} = \sup_x \Pr\{x \leqq \xi_1 + \ldots + \xi_n \leqq x+L\},$$

$$s = \sum_{k=1}^{n} [1 - Q_k(l)].$$

If $L \geqq l$, then

$$Q(L) \leqq \frac{CL}{l\sqrt{s}},$$

where C is an absolute constant.

(B. A. Rogozin)

2/474

GJEDDEBAEK, N. F. (A/S Ferrosan, Copenhagen) **3.6** (4.3)

Contributions to the study of grouped observations.
IV. Some comments on simple estimates—*In English*

Biometrics (1959) **15**, 433-439 (10 references, 1 table)

This paper is a continuation of a series by the author: previous publications were in *Skandinavisk Aktuarie-tidskrift* (1949) **32**, 135; (1956) **39**, 154 and (1957) **40**, 20. This present paper investigates the means and variances of simple estimators of the parameters of a normal distribution with grouped observations and where *a priori*

(i) one parameter is known
(ii) both parameters are unknown.

In the first case where σ is known, one estimator of μ is compared with another and the usual interpretation given to the variance of either estimator. Because the simple estimators of μ and σ^2 are inconsistent, the usual concept of efficiency cannot apply. The author proposes a derivation of this concept for his particular purpose and shows that the inconsistent estimators of μ and σ^2 have zero asymptotic efficiency. Analogous reasoning applies to the situation where μ is known.

Where both parameters are unknown it is shown that where the grouping is not equidistant the procedure to use is that of maximum likelihood: otherwise Sheppard's correction may be used with a simple estimator based upon $N-1$ degrees of freedom. It is concluded that, for equidistant grouping with intervals less than twice the standard deviation, the simple estimator of the variance completed with Sheppard's correction is as efficient practically as the maximum likelihood estimator at least for a number of observations less than one hundred.

(W. R. Buckland)

2/475

GJEDDEBAEK, N. F. (A/S Ferrosan, Copenhagen) **3.6** (4.3)

Contribution to the study of grouped observations. V. Three-class grouping of normal observations—*In English*

Skand. Aktuartidksr. (1959) **42**, 194-208 (10 references, 1 table, 21 figures)

In three previous articles [*Skand. Aktuartidskr.* (1949) **32**, 135; (1956) **39**, 154; (1957) **40**, 20; and *Biometrics* (1959) **15**, 433-439; abstracted in this present journal No. **2**/475, **3.6**] the author has dealt with the efficiency of estimates of the mean and standard deviation obtained from grouped normal observations. The present paper is mainly confined to a detailed numerical investigation of a particular case.

A sample of N is drawn from a normal population with mean ξ and standard deviation σ. The sample is grouped into three classes with the fixed boundaries $-\infty < D_1 < D_2 < \infty$. The corresponding class frequencies are n_1, n_2 and $n_3 = N-n_1-n_2$. The normal equivalent deviates of n_1/N and $(n_1+n_2)/N$ are Y_1 and Y_2 respectively. It has been shown that, when only the three class-frequencies are given, the maximum likelihood estimates of σ and ξ are

$$S = (D_2-D_1)/(Y_2-Y_1)$$
$$M = D_1-Y_1S = D_2-Y_2S.$$

Let $m = (M-\xi)/\sigma$, and $t_N = \sqrt{N}(M-\xi)/S$. As N tends to infinity, $m\sqrt{N}$ and t_N tend to be normally distributed with the same standard deviation, τ (say). A table of τ is given for different values of x_1 and x_2, where $x_i = (D_i-\xi)/\sigma$.

The exact distributions of m and $t_N/\sqrt{N}$ have been computed in the case $N = 40$ for 21 different combinations of x_1 and x_2. The results are shown graphically. It is found that $t_N/\sqrt{N}$ is much closer to normality than is m. The agreement between the distribution of $t_N/\sqrt{N}$ and the limiting form is judged to be satisfactory when each n_i has an expectation not less than five.

Rules for a confidence interval for ξ are given. They involve an iterative procedure for finding the values of τ to be used in the limits

$$M-t\tau_1S/\sqrt{N}, \qquad M+t\tau_2S/\sqrt{N}$$

(t is the normal deviate corresponding to the desired confidence level).

(B. Matérn)

2/476

MALLOWS, C. L. (University College, London; now Columbia University, N.Y.) **3.6** (–.–)

On the moments of small samples—*In English*

Trab. Estadíst. (1960) **11**, 119-121 (1 reference)

In this paper the author states that too small a sample cannot give close estimates of all the lower moments of the population simultaneously. Suppose we have, for instance, a sample of size $n = 4$ from a normal population: then the first four raw-moment estimators m'_r, have expectations 0, 1, 0, 3 respectively, but there do not exist four real numbers x_i such that $m'_r(x) = \mu'_r(x)$ for $r = 1, 2, 3, 4$. The lowest n for which these estimators can take these values is $n = 6$ and there is just one possible sample which can have zero probability. For $n \geqq 7$ there is a positive probability density at the expectation point.

If we are using k-statistics to estimate the cumulants with a normal population, if $n \geqq 4$, then $k_1, k_2, \ldots, k_4$ can simultaneously take their expected values; if $n \geqq 6$, then this applies to $k_1, \ldots, k_6$ and if $n \geqq 9$ it applies to $k_1, \ldots, k_8$. For the central-moment estimators, the figures are $n \geqq 5$ for four moments and $n \geqq 13$ for six moments.

These results are obtained from the polynomial $Q(x)$ of degree n whose zeros are the observations $x_1, \ldots, x_n$; it has coefficients which are elementary symmetric functions of the x_i.

(R. Pro)

2/477

ROSSBERG, H. J. (Berlin) **3.8** (1.5)

On the distributions of the differences and ratios of order statistics—*In German*

Math. Nachr. (1960) **21**, 37-79 (3 references)

Let $x_1, x_2, \ldots, x_n$ be independent, one-dimensional random variables with the same distribution function $\Pr\{x_i < x\} = F(x)$. The related order statistics are designated by $\xi_1, \xi_2, \ldots, \xi_n$. The present paper studies the distributions of the differences $d_{kh} = \xi_k - \xi_h$ and the ratios $q_{kh} = \xi_h/\xi_k$ and is especially devoted to limit theorems ($n \to \infty$) and to questions of statistical dependence or independence. The research is based on results of Gnedenko and Smirnoff on the limit distribution of $\Pr\{\xi_k < a_n x + b_n\}$.

It is proved that three types of distributions can occur as the limit distribution of $\Pr\{d_{kh} < a_n x\}$ with suitable constants $a_n > 0$. Sufficient conditions are given for the occurrence of these three types.

Analogous limit theorems are proved for the joint distribution $\Pr\{d_{lh} \geqq a_n x,\ d_{mk} \geqq a_n y\}$. The following five cases are treated separately:

(i) $h < k = l < m$

(ii) $h < l < k < m$

(iii) $h = k < l < m$

(iv) $h < k < l = m$

(v) $h < k < l < m$.

An investigation to discover in which cases the differences d_{kh} and d_{ml} $(h < k \leqq l < m)$ are independent or dependent in the limit is carried out.

The sufficient conditions mentioned above concerning $\Pr\{d_{kh} < a_n x\}$ turn out to be sufficient and necessary for all linear combinations $c_1\xi_1 + \ldots + c_k\xi_k$ $(c_k \neq 0)$ to have non-degenerate limit distributions for fixed k if suitably normalised. All possible types of limit distributions for $\Pr\{q_{kh} < \gamma_n x + \delta_n\}$, with suitable constants $\gamma_n > 0$ and δ_n and continuous $F(x)$, are derived. Again, sufficient conditions are given. The author specifies which of the conditions is fulfilled if the original distribution $F(x)$ is either the normal distribution, the t-distribution or any other of the well-known distributions.

In a final chapter the statistical dependence of the order statistics ξ_1, the differences d_{kh}, the ratios q_{kh} and certain functions of these variables are explored. For example, the vectors $(\xi_1, \ldots, \xi_{l-1})$ and $(\xi_{l+1}, \ldots, \xi_n)$ are independent on the condition that $\xi_l = a\,(1 < l < n)$. For the independence of d_{kh} and ξ_l and likewise of q_{kh} and ξ necessary and sufficient conditions are set up. In addition sufficient conditions for the independence of all differences and likewise of all ratios are given.

(W. Uhlmann)

2/478

ROSSBERG, H. J. (Inst. Appl. Math. & Mech., German Academy of Sciences, Berlin) **3.8** (1.5)
Some limiting theorems for order statistics—*In German*
Second Hungarian Mathematical Congress (1961)

Let $x_1, \ldots, x_n$ be n independent random variables with common distribution function $\Pr\{x_i < x\} = F(x)$. Let us denote the corresponding order statistics by $\xi_1, \ldots, \xi_n$. Under these conditions the investigations concerning the limiting distribution of the order statistics ξ_k for $n \to \infty$ are essentially completed by the papers of Gnedenko (1943) and Smirnoff (1949), and this is done for " external numbers "; k = const. or $n-k$ = const., and also for " central members " ($k = k_n \to \infty$, $k/n \to \lambda$, $0 < \lambda < 1$). In particular the extremal members turned out to have the class of possible limiting distributions consisting of three proper types of distribution which are still dependent on parameters.

On the basis of the results of the authors noted above the limiting distributions of the differences $\xi_k - \xi_h$ $(k > h)$ may be investigated. It turns out for k = constant and h = constant, if the limiting distribution of ξ_k exists, then the same holds for $\xi_k - \xi_h$ and this can be given explicitly. Concerning the question as to whether the class of the possible limiting distributions of $\xi_k - \xi_h$ is exhausted, some statements can be made which, however, do not completely solve the problem. Under the same condition also the limiting distributions of the vector $(\xi_k - \xi_h, \xi_s - \xi_t)$ $(k > h, s > t)$ exist. In some cases limiting independence between $\xi_k - \xi_h$ and $\xi_s - \xi_t$ appears if $k \leq t$ in the sense that the limiting distribution consists of the product of two distribution functions, for which there are given necessary and sufficient conditions. Even under the supposition of h = constant, $k/n \to \lambda$ $(0 < \lambda < 1)$ and that of the existence of the limiting distributions of ξ_h and ξ_k, $\lim \Pr(\xi_k - \xi_h < a_n x + b_n)$ may be calculated for appropriate sequences $\{a_n\}$ and $\{b_n\}$. Concerning this question the conditions assuring the limiting independence of the order statistics are of interest. This is known to be the case for the extremal members ξ_h and ξ_k given h = constant and $n-k$ = constant; according to Geffroy [*Publ. Inst. Statist. Paris* (1959) **8**, 36-121, 124-185; abstracted in this journal **2**/253, **2.6**] and a subsequent paper in 1960, even in a very strong sense. The limiting independence, however, holds between extremal and central members too; if h is constant then $k \to \infty$.

Further investigations concern the necessary and sufficient conditions for the stability of $\sum_1^l c_i \xi_i$; where l is a constant and c_l different from zero.

(H. J. Rossberg)

2/479

SRIVASTAVA, A. B. L. (Statistical Lab., Indian Inst. Tech., Kharagpur) **3.5** (2.7)
The distribution of regression coefficients in samples from bivariate non-normal populations. I. Theoretical investigation—*In English*
Biometrika (1960) **47**, 61-68 (8 references)

The author assumes a bivariate population which may be represented by some terms of a double Edgeworth series and considers the distribution of the regression coefficient in samples of N drawn from such a population.

The technique used is that put forward by Gayen. The distribution of the regression coefficient is derived and the statement is made that this distribution holds when the population correlation coefficient is not necessarily small; and for any population provided the sample size N is large. It will also hold when the sample size N is small for populations for which the fifth and higher order cumulants are negligible.

The distribution of the t-statistic used for testing the significance of the regression coefficient is also derived under the Edgeworth population assumption both for the case of known and of unknown marginal variances. The author states that he intends to carry out a numerical investigation.

(Florence N. David)

2/480

VINCZE, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **3.8** (5.3)

On the deviation of two-variate empiric distribution functions—*In Hungarian*

Magy. Tud. Akad. III Oszt. Közl. (1960) **10**, 361-372 (9 references)

After summarising the most important one-sample and two-sample tests which are based on the deviations of the distribution functions and mentioning the result of Kiefer & Wolfowitz [" On the deviations of the empiric distribution function of vector chance variables", *Trans. Amer. Math. Soc.* (1958) **87**, 173-186] the author gives theorems concerning the deviations of the empiric distributions of two two-dimensional variates.

Let (ξ_i, η_i), $i = 1, 2, \ldots, N$ be continuous two-dimensional variates independently and identically distributed with distribution function $F(x, y)$. Similarly the independent and identically distributed two-dimensional variates (ξ'_i, η'_i), $i = 1, 2, \ldots, N$ have the distribution function $G(x, y)$. Let $F_N(x, y)$ and $G_N(x, y)$ denote the corresponding empirical distribution functions. Let

$$D^+_{NN}(y) = \max_{(x)} [F_N(x, y) - G_N(x, y)]$$

$$D_{NN}(y) = \max_{(x)} | F_N(x, y) - G_N(x, y) |$$

Let us choose the value of $\eta = y$ randomly according to the probability law $F(\infty, y)$. Let $P^{(k)}_{NN}$ and $P^{|k|}_{NN}$ denote the probabilities that $D^+_{NN}(y)$ and $D_{NN}(y)$ do not exceed k/N, respectively. The author's theorems give the exact probabilities $P^{(k)}_{NN}$ and $P^{|k|}_{NN}$ and also their limiting values.

(K. Sarkadi)

2/481

ŽALUDOVÁ, Agnes H. (Heat Engineering Research Institute, Prague) **3.2** (3.6)

Note on the non-central *t*-test based on range—*In English*

Second Hungarian Mathematical Congress (1961)

The non-central t statistic based on a root-mean-square estimate of the population standard deviation σ was dealt with in the original paper of Johnson & Welch [*Biometrika* (1940) **31**, 362-389]. For purposes of practical application, especially in sampling inspection by variables, it is convenient to define the non-central statistic q using the standard deviation estimate based on mean sample range. Tables for the percentage points q_ε computed for the exact sampling distribution of mean range have been published recently by the present author [*Acta Tech., Czech. Acad. Sciences* (1960)].

Since it is known that the distribution of mean range in m independent samples each of size n can be approximated very well by the distribution $c\chi_\nu\sqrt{\nu}$, where c and ν have been tabulated as functions of m and n by Patnaik [*Biometrika* (1950) **37**, 78-87], it is possible to make several comparisons of the directly computed values q_ε with values obtained using the above approximation. Three such comparisons have been made:

(i) based on the original tables of Johnson and Welch,

(ii) based on the Pearson Type IV curve approximation to the non-central t-distribution suggested by Merrington & E. S. Pearson [*Biometrika* (1958) **45**, 484-491; abstracted in this journal **1**/39, **3.2**],

(iii) based on the tables of the non-central t-distribution published by Resnikoff & G. J. Lieberman (1957) Stanford.

Agreement of the results is discussed.

(Agnes H. Žaludová)

2/482

CHAPMAN, D. G. & ROBSON, D. S. (Univ. Washington, Seattle and Cornell Univ., N.Y.) **4.3** (2.5)

The analysis of a catch curve—*In English*

Biometrics (1960) **16**, 354-368 (10 references, 1 table)

The age distribution of a random sample, or the " catch curve " from a stationary animal or fish population provides information on the annual survival rate s of the population. It is commonly assumed that there is some age x_0, such that for all ages $x \geqq x_0$, the probability of selection is the same and the annual survival rate is the same. If ages are relabelled so that $x_0 = 0$ the age distribution of the population is geometric, that is to say,

$$f(x) = as^x, \quad x = 0, 1, 2, \ldots,$$

where $a = 1-s$ is the annual mortality rate.

The best (uniformly minimum variance unbiased) estimate of s is $\hat{s} = \bar{x}/(1+\bar{x}-1/n)$, $\bar{x}$ being the mean age of the sample. The best estimate of Var $\hat{s}$ is $\hat{s}[\hat{s}-(t-1)/(n+t-2)]$ with $t = n\bar{x}$.

While the model giving rise to the geometric age distribution may seem artificial, its plausibility is greater than might appear since, as the authors show, it can be regarded as the result of a random birth and death process when suitable assumptions are made.

The estimate $\hat{s}$ is compared to the so-called " Jackson " estimate and some modifications thereof. Assumptions basic to regression estimates of the survival rate are considered.

Sometimes the sampling gear is selective against older ages; also, the assumption of a constant survival rate may be valid for only a limited number of age classes. In these cases the age distribution may be taken to be a truncated geometric over a reduced number of age classes. Tables facilitating the estimation of s by maximum likelihood are provided.

A test of the basic model is given for the particular case where it is suspected that there may be selection against younger ages. Finally, the authors prove that for the geometric model there is no unbiased estimate of the instantaneous mortality rate $i = -\log s$, but that a nearly unbiased estimate is given by

$$i^* = \log(1+\bar{x}-1/n) - \ln \bar{x} - (n-1)(n-2)/n(t+1)(n+t-1).$$

(H. A. David)

2/483

COHEN, A. C., Jr. (University of Georgia, Athens, U.S.A.) **4.3** (2.3)

Simplified estimators for the normal distribution when samples are singly censored or truncated—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 251-269 (11 references, 2 tables)

In life-testing, dosage response studies, target analyses, biological assays and in other related investigations the selection and observation of variables is often restricted over some portion of the range of possible population values; that is to say for some values of the random variable involved. Depending on the nature of the restriction, samples obtained in this way are designated as truncated or censored. Samples in which certain population values are entirely excluded from observation are described as truncated. Those in which sample specimens with measurements falling in certain restricted intervals of the random variable may be identified and thus counted but not otherwise observed, while the remaining sample specimens may be observed without restriction, are described as censored. The present paper deals with maximum-likelihood estimation of the mean and variance of a normally distributed population from samples of these types.

The estimators obtained in this paper are simpler than those previously given for the cases considered here, in that the mean and variance of the population are estimated by the mean and variance of the sample plus simple corrections. These corrections involve only one auxiliary function and the practical application of these estimators consequently necessitates interpolation in only one table, and not in two or more, as was formerly necessary.

For the case of singly-truncated or singly-censored samples the auxiliary functions are approximately linear if the intervals are not too wide, so that exact interpolation between the number in the table is relatively easy in both cases.

(A. C. Cohen, Jr.)

EPSTEIN, B. (Wayne State Univ., Detroit and Stanford Univ., Cal.) **4.3** (2.5)
Estimation from life test data—*In English*
Technometrics (1960) **2**, 447-454

In this paper four estimation procedures are discussed which can be used to make point or confidence estimates from life test data. The first two procedures can be applied when the underlying density of life is exponential: the third and fourth are non-parametric.

If life testing is discontinued after a fixed number, r, of items have failed, where items on test may or may not be replaced and where the number of items initially on test is $n \geqq r$, the " best " estimate of mean-life, θ, is given by $\hat{\theta}_{r,n} = T_{r,n}/r$ where $T_{r,n}$ is the accumulated life on test until the rth failure occurs. Furthermore $100(1-\alpha)$ per cent. confidence intervals for θ can be found using the chi-squared distribution. Similarly, expressions are available where the life test is discontinued after a fixed amount of total life T has elapsed.

Two non-parametric procedures are outlined. In one case n items are tested for time t^* and the number of items that fail in $[0, t^*]$ are counted; the items that fail are replaced. In the other case, items are drawn at random from some population and tested one at a time for a length of time t^*. Testing continues until a preassigned number r have failed. In both cases, one-sided confidence interval estimates of both the reliability and the parameter θ are given.

(P. H. Randolph)

2/485

GRAYBILL, F. A. & MORRISON, R. D. (Oklahoma State University, Stillwater) **4.4** (3.6)
Sample size for a specified width confidence interval on the variance of a normal distribution—*In English*
Biometrics, (1960) **16**, 636-641 (2 references, 1 table)

This paper is concerned with the determination of the sample size needed for the confidence interval estimation of a parameter when the length of the interval is specified. The authors cite the general problem that confronts the experimenter and then proceed to illustrate a method for solving this problem, the specific parameter being the variance of a normal distribution.

The determination involves the following assumptions: (*a*) The probability that the confidence interval contains the parameter is pre-selected; (*b*) The probability that the width of the confidence interval will be less than a specified quantity is greater than or equal to a given value. The authors refer to these two probabilities as the confidence coefficient and width coefficient respectively.

The problem of determining the sample size in order that (*a*) and (*b*) are satisfied is outlined in detail. The explanation is supplemented with a set of tables which is needed in the determination. The theory involved is not presented. Assuming an unbiased estimate of the variance is known, the sample size is found by entering into the tables the following values: (1) (*a*) and (*b*) above; (2) An expression to be computed involving the specified confidence interval, the chi-square variable, the number of observations on which the variance estimate was based, and the estimate itself. The degrees-of-freedom can then be read, and hence the sample size obtained. The tables are presented for various values of the significance level and width level. An example illustrating their use is also given.

The authors point out that these tables can also be used to solve a similar problem. If it is desired that the width of the confidence interval be a particular proportion of the variance itself, the required sample size can be determined. As usual, the confidence coefficient and width coefficient are chosen. The definition of the latter must however be altered to fit the desired pre-selected conditions. The method for the determination is outlined by the authors and an example is given.

(R. H. Myers)

2/486

GITTELSOHN, A. M. (State Department of Health, Albany, N.Y.) **4.3** (2.4)

A model for the analysis of the distribution of qualitative characters in sibships—*In English*

Biometrics (1960) **16**, 534-546 (5 references, 1 table)

The author considers the problem of estimating frequencies of recessive characters through sibship data. The basic treatment which assumed fixed sibship size and a binomial distribution for the number of recessives from the sibship is reviewed. A general model is introduced with potential sibship size as a random variable. The model is used to describe situations where family size is determined by the composition of sibs produced as well as natural size variability arising out of differences in fertility between couples.

Each couple is thought of as possessing a potential or ideal family size, jointly mediated by fecundity and the desire for a particular number of children. The realised number of children produced will then depend upon the couple's success in attaining their ideal family size. Failure may be through loss of fertility, faulty contraceptive technique or through a downward revision in their ideal family size as a result of having produced one or more children affected by the condition studied. In the latter circumstances, such selective termination insures that realised family size will depend on family composition.

Two types of ascertainment methods are considered. " Complete ascertainment through affected children " whereby all sibships with at least one affected child are observable. The probability of ascertaining a sibship with at least one affected is therefore taken as certainty. The second method, " incomplete multiple ascertainment through affected children " whereby sibships come under observation with probability dependent upon the number of affected children and may be ascertained as many times as there are affected children.

Representing sibship size as a geometric and as a Poisson variable, and the number affected and the number of independent ascertainments of a given sibship as binomials, maximum likelihood estimates of the parameters are obtained for the case of no selective termination of reproduction. Both methods of ascertainment are considered. Using the same model, maximum likelihood estimates are obtained for the case

continued

GITTELSOHN, A. M. (State Department of Health, Albany, N.Y.) **4.3** (2.4)

A model for the analysis of the distribution of qualitative characters in sibships—*In English*

Biometrics (1960) **16**, 534-546 (5 references, 1 table)

continued

of termination of reproduction at the birth of the first affected r ($r = 1, 2, \ldots$) children. The model is also extended to populations where the termination rule may vary among couples.

Estimates of the probability that a child is affected with the condition under study and the probability that an affected child is detected are given for both ascertainment method and termination rule. The effect of various assumptions in the model can be readily observed.

(G. Krause)

JÍLEK, M. & LÍKAŘ, O. (Central Res. Inst. Food Industr. and Res. Inst. Food Industr. Econ., Prague) **4.5** (2.3)

Tolerance regions of the normal distribution with known μ, and unknown σ—*In English*

Biom. Zeit. (1960) **2**, 204-209 (6 references, 4 tables)

In this paper the tolerance coefficients k_1, k_2 of a normal distribution $N(\mu, \sigma)$ with known μ, unknown σ for the determination of two- and one-sided tolerance regions $\mu \pm k_1 s$ and $\mu +$ or $-k_2 s$ respectively are defined, in such a way that at least a given proportion p of the distribution lies within the limits with given probability β. The coefficients are given in tables to 3 significant decimal places for sample sizes $n = 3\,(1)\,20\,(5)\,50\,(10)\,100$ and up to 10000 and ∞, $p = 0{\cdot}90$; $0{\cdot}95$; $0{\cdot}975$; $0{\cdot}99$; $0{\cdot}995$; $0{\cdot}999$ for two-sided limits (correspondingly $p' = 2p-1$ for one-sided limits) and $\beta = 0{\cdot}90$; $0{\cdot}95$; $0{\cdot}99$; $0{\cdot}999$.

(R. Wette)

KIMBALL, A. W. (Oak Ridge Nat. Lab., Tennessee) **4.6** (10.9)

Estimation of mortality intensities in animal experiments—*In English*

Biometrics (1960) **16**, 505-521 (17 references, 2 tables, 1 figure)

This paper discusses nonparametric methods for estimating mortality intensities from small samples in controlled experiments uncomplicated by losses and age variations, and compares them with respect to their usefulness in interpreting radiation mortality data. Particular emphasis is placed on the construction of Gompertz plots, that is to say a semi-logarithmic graph of the age-specific death rates. Some of the techniques have been published in journals which are relatively unfamiliar.

Four procedures of estimation are discussed. Starting with a random sample of individuals from a population with an unknown distribution function of age at death, the first method is based on preassigned time intervals while the second is based on preassigned numbers of deaths. For the first procedure time intervals of equal lengths are chosen arbitrarily and the number of deaths in a particular interval are considered to be random variables. For the second procedure the number of deaths are fixed in advance of the experiment and the time intervals are considered to be random variables.

The biases of the estimate in classical actuarial use and of the estimate which assumes mortality intensity to be non-decreasing are obtained for the preassigned time interval procedure. A Gompertz-Makeham mortality function was fitted to real data and the estimates from this fit were used as parameters in a numerical study of bias. For the preassigned numbers of deaths, the procedure proposed by Seal [*Skand. Aktuartidskr.* (1954) **37**, 137-162] is given and a modification is proposed.

An example illustrating the four methods of estimation is given. A Gompertz plot is included for the data and an appendix deriving the asymptotic variances is also included.

(C. Y. Kramer)

OLIVEIRA, J. T. DE (University of Lisbon) **4.8** (4.3)

Bivariate distributions with given marginal distributions and the estimations of their parameters—*In French*

Second Hungarian Mathematical Congress (1961)

In some problems concerning bivariate continuous random variables, the marginal distributions $F(x, +\infty)$ and $F(+\infty, y)$ are known; in other cases the marginal distribution is known with the exception of some parameters, for example the location and dispersion parameters. The estimation of these parameters from the samples of X's or Y's gives, in general, asymptotically the same results. The first case is considered.

The uniform transformation $U = F(X, +\infty)$, $V = F(+\infty, Y)$ gives a random vector having the distribution function $K(u, v)$ with

$K(u, 0) = K(0, v) = 0, \quad K(1, v) = v \quad \text{and} \quad K(u, 1) = u.$

Thus the density $k(u, v)$ may take the form $k(u, v) = 1+h(u, v)$ where $h(u, v)$ depends on two almost completely arbitrary functions. Therefore the estimation of the parameters of $K(u, v)$ is non-parametric for the marginal distributions.

The parameter functionals

$$P(K) = \int_{-\infty}^{\infty} \cdots \int g(u_1, v_1; \ldots; u_k, v_k) dK(u_1, v_1) \ldots dK(u_k, v_k)$$

can be estimated by

$$\binom{n}{k}^{-1} [g(u_{\alpha_1}, v_{\alpha_1}; \ldots; u_{\alpha_k}, v_{\alpha_k})] - a$$

method due to Hoeffding, the asymptotical distributions of which are multinormal; the same holds for the functionals, the first derivatives of which do not vanish for K.

Suppose, for sake of simplicity, that $K(u, v)$ has only two parameters θ_1 and θ_2. One may find two functions $g_1(u, v)$ and $g_2(u, v)$ the mean values of which are functions of θ_1 and θ_2 independent of the parameters θ_2 and θ_1 respectively. Therefore, the asymptotic distributions of the estimators of θ_1 and θ_2 obtained by the inversion of the estimators of $M(g_1)$ and $M(g_2)$ are bi-normal.

(J. T. de Oliveira)

SCHMETTERER, L. (University of Hamburg) **4.1** (4.0)

On the theory of unbiased estimators—*In German*

Second Hungarian Mathematical Congress (1961)

The problem of the existence and characterisation of local minimal unbiased estimates is dealt with; the question of their existence under certain suppositions is solved by means of the following theorem.

The norm of a ratio space relative to a closed linear subspace of a Banach-space is always taken on if the subspace is reflexive. Local minimal estimators can be characterised under very general conditions by the vanishing of a Gateaux differential. With the aid of this characterisation the minimal estimators can be determined explicitly in some cases; in order to do this a known lemma concerning the representation of linear functionals in a ratio space is needed.

(L. Schmetterer)

SHUMWAY, R. & GURLAND, J. (Iowa State Univ., Ames, U.S.A.) **4.3** (2.5)

Fitting the Poisson binomial distribution—*In English*

Biometrics (1960) **16**, 522-533 (7 references, 4 tables)

The authors have developed simplified methods for computing probabilities and for obtaining maximum likelihood estimates of the parameters, a and p, in the Poisson binomial distribution

$$\Pr(k) = e^{-a} \sum_{t=0}^{\infty} \frac{a^t}{t!} \binom{nt}{k} p^k q^{nt-k}.$$

A table is provided which greatly simplifies computation.

The density function can be written to contain explicitly the expression for the kth factorial moment, $\mu_{[k]}$, of the variable nX, where n is a known positive integer and X is Poisson with mean λ, $\lambda = aq^n$, and $q = 1-p$. Exploiting this device, a recurrence relation for the computation of $\Pr(k)$ is

$$\Pr(k+1) = \frac{p}{q} \left\{ \frac{\Pr(k)}{k+1} \right\} p_{[k]},$$

where

$$p_{[k]} = \frac{\mu_{[k+1]}}{\mu_{[k]}}.$$

Given values for $p_{[k]}$, the computation is obviously simple. The likelihood equations are $n\hat{a}\hat{p} = \bar{k}$ and

$$L(\hat{p}) = \frac{1}{n\hat{a}q} \Sigma\, a_k p_{[k]} - N,$$

where the a_k are observed frequencies and N is the sample size.

The first estimate of p is made by some simple method; for example, sample moments or frequencies—and it is then calculated accurately by application of Newton's method, using $L(\hat{p})$ and a modification $L'(\hat{p})$. Obtaining $\hat{a}$ once $\hat{p}$ is known is straightforward.

As in computation of the probabilities, the procedure is not difficult if the $p_{[k]}$ and, in this case, the $q_{[k]}$ are known. The authors have computed the values for $p_{[k]}$ and $q_{[k]}$ for different values of λ, $\lambda = 0{\cdot}10$ $(0{\cdot}02)$ $1{\cdot}10$ and for $k = 0\,(1)\,9$. Their methods for computing $p_{[k]}$ and $q_{[k]}$ are developed and exemplified in one section of the paper.

Two examples are given showing the estimation of Poisson binomial probabilities and parameters. Some comments are made on expedient means of interpolation in the table $p_{[k]}$, $q_{[k]}$; there is also a brief discussion of the problem which arises when the zero class is missing, that is to say when the distribution is truncated on the left.

(H. R. Baird)

2/493

SPECHT, W. (Universität Erlangen) **4.8** (4.0)

On the theory of elementary mean values—*In German*

Math. Zeit. (1960) **74**, 91-98 (1 reference)

Let P_n be the set of all n-dimensional, real vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with positive components x_i and $n > 1$ and let $p = (p_1, p_2, \ldots, p_n)$ be a distribution of positive weights p_i with $p_1 + p_2 + \ldots + p_n = 1$. The elementary mean value of the vector $\mathbf{x}$ is defined by

$$M_s(p, \mathbf{x}) = (\Sigma\, p_r x_r^s)^{1/s}$$

for each real number $s \neq 0$ and

$$M_0(p, \mathbf{x}) = \exp(\Sigma\, p_r \log x_r).$$

The limits for s to $-\infty$ and s to $+\infty$ are

$$M_{-\infty}(p, \mathbf{x}) = \min x_r \quad \text{and} \quad M_{+\infty}(p, \mathbf{x}) = \max x_r.$$

Let $Q_{s,t} = Mt/Ms$, then it is wellknown that $Q_{s,t}(p, \mathbf{x}) \geqq 1$ for all s and t with $s < t$.

$Q_{+\infty, -\infty} = \max x_r / \min x_r$ is called the relative range $\beta(\mathbf{x})$ of the vector $\mathbf{x}$. For all vectors $\mathbf{x}$ of P_n with $1 \leqq \beta(\mathbf{x}) \leqq B$ and all distributions p of weights, an upper bound for $Q_{s,t}(p, \mathbf{x})$ is given for $-\infty < s < t < +\infty$. This upper bound depends on t, s and B only and is proved to be optimal.

Finally this research is extended to functions $f(x)$, which are defined, positive and integrable in the interval $[0, 1]$. Let $p(x) \geqq 0$ be defined in the interval $[0, 1]$ with the integral of $p(x)$ between the limits 0 and 1 equal to 1. The integral average of $f(x)$ is defined by

$$M_s(p, f) = \left[\int_0^1 f^s(x) p(x) dx \right]^{1/s} \quad \text{for } s \neq 0$$

and

$$M_0(p, f) = \exp\left[\int_0^1 \{\log f(x)\} p(x) dx \right].$$

The definition of $Q_{s,t}(p, f)$ is analogous to the definition given above. For all functions $f(x)$ integrable in $[0, 1]$, for which

$$0 < a \leqq f(x) \leqq A, \quad \text{for } 0 \leqq x \leqq 1$$

the same upper bound of $Q_{s,t}(p, f)$ as above is valid for $-\infty < s < t < +\infty$ if B is replaced by A/a.

(W. Uhlmann)

BAHADUR, R. R. (Indian Statistical Institute, Calcutta) **5.1** (1.5)

Stochastic comparison of tests—*In English*

Ann. Math. Statist. (1960) **31**, 276-295 (13 references)

Let $T_n^{(1)}$ and $T_n^{(2)}$ be two sequences of real-valued test statistics for testing the hypothesis that θ belongs to a sub-set Ω_0 of Ω, where Ω indexes a set of probability distributions.

The method of stochastic comparison outlined below gives a measure of the relative efficiency of the two sequences at each parameter point in $\Omega-\Omega_0$. It is assumed that each sequence has a limiting distribution function independent of θ in Ω_0, and for θ in $\Omega-\Omega_0$, $T_n^{(i)}/\sqrt{n}$ converges in probability (P_θ) to a positive number $b_i(\theta)$. Let $K_n^{(i)} = -2\log[1-F_i(T_n^{(i)})]$, and let $C_i(\theta)$ be the limit in probability (P_θ) of $K_n^{(i)}/n$. Under certain assumptions made on F_i, $C_i(\theta) = a_i[b_i(\theta)]^2$ for θ in $\Omega-\Omega_0$, where $a_i>0$ depends on the distribution function F_i. The ratio $\phi_{12}(\theta) = C_1(\theta)/C_2(\theta)$ is proposed as a measure of efficiency of the first sequence relative to the second sequence. The author gives some justification for this as a measure of efficiency on intuitive grounds and shows that in cases where the Pitman theory of efficiency is applicable, the Pitman efficiency is obtained as a limit of $\phi_{12}(\theta)$ as θ approaches some point in Ω_0.

Furthermore, it is shown that, if for all n the power of the test $T_n^{(1)}$ never exceeds the power of $T_n^{(2)}$ at θ, $\phi_{12}(\theta)\leqq 1$; and conversely, if $\phi_{12}(\theta)<1$, then the above inequality must hold for each n. In addition, it is shown that under certain conditions $C(\theta)$ can be represented as the limit as n tends to infinity of $1/n$ times the expected ratio of the power of T_n to the size of T_n where the size is chosen at random according to a certain fixed distribution.

Several examples are given in which the relative efficiencies are computed for standard tests of the following hypotheses: the mean of a normal distribution is zero, the variance of a normal distribution is one, two distribution functions are equal, several distribution functions are equal, and that a bivariate normal distribution has correlation coefficient zero.

(D. R. Truax)

2/495

BENNETT, B. M. & HSU, P. (University of Washington, Seattle) **5.2** (6.9)

On the power function of the exact test for the 2×2 contingency table—*In English*

Biometrika (1960) **47**, 393-397 (9 references, 10 figures)

The exact tests of significance in a 2 by 2 table for the equality of two population proportions is discussed: this hypothesis was set out by Fisher in 1935. The conditional power function under the hypothesis that these proportions are not equal and also the overall power functions are discussed.

Ten power diagrams for different sample sizes have, been calculated: (5, 5), (10, 10), (10, 5), (15, 15), (15, 10) (15, 5), (20, 20), (20, 15), (20, 10) and (20, 5). A comparison with Sillitto's approximation is made.

(Florence N. David)

2/496

BHATE, D. H. (University Coll., London and Delhi School of Economics) **5.7** (5.2)

Approximation to the distribution of the sample size for sequential tests. II. Tests of composite hypotheses—*In English*

Biometrika (1960) **47**, 190-193 (16 references)

The author extends the procedure of his previous paper [" Approximation to the distribution of the sample size for sequential tests. I. Tests of simple hypotheses ", *Biometrika* (1959) **46**, 130-139: abstracted in this journal No. **1**/223, **5.7**] to the case of composite hypotheses.

Assume two normal populations with means μ_1 and μ_2 and variances σ_1 and σ_2 respectively with $\lambda = \sigma_1^2/\sigma_2^2$. The hypothesis H_0 is $\lambda = \lambda_0$ and H_1 that $\lambda = \lambda_1$. Given that at a certain stage there are samples, each of m measurements, available from each population, the author suggests the calculation of the quantity F_m which is the ratio of the two sample sums-of-squares and proposes the rules:

(i) continue to the $(m+1)$st stage if $F_m^- < F_m < F_m^+$,

(ii) accept H_0 if $F_m^- \geqq F_m$,

(iii) accept H_1 if $F_m^+ \leqq F_m$.

F_m^- and F_m^+ are the solutions, respectively, of the equations

$$\beta/1-\alpha,$$
$$(1-\beta)/\alpha = (\lambda_0/\lambda_1)^{\frac{1}{2}(m-1)}\{(1+F_m/\lambda_0)/(1+F_m/\lambda_1)\}^{m-1}$$

where α and β have the meanings generally given in sequential analysis. At the end of this note some examples are given.

(Florence N. David)

2/497

BROSS, I. D. J. (Roswell Park Memorial Inst., Buffalo, N.Y.) **5.0** (11.9)

Statistical criticism—*In English*

Cancer (1960) **13**, 394-400 (13 references, 2 tables)

This is a discussion of the role and responsibility of the statistical critic. The author feels that an essential feature of statistical criticism is the implicit or explicit presentation of a hypothesis counter to the one being criticised.

The following tentative criterion for criticism is suggested. " The critic has the responsibility for showing that his counter hypothesis is tenable. In so doing, he operates under the same ground rules as a proponent."

Although this criterion would require proponent and critic to operate under the same rules in establishing their respective hypotheses, the burden of proof would rest on the proponent. The proponent would have to rule out all tenable hypotheses except his own, the critic need only show his hypothesis is tenable. The recommended minimum requirement for the establishment of tenability by the critic is that the effects predicted from his hypothesis be in line with the actual data both in direction and order of magnitude. The amount of additional argument required would depend on the hypothesis.

Discussion and examples of several types of criticism constitute the bulk of the paper. The topics treated are

(1) " Hit and run " criticism (enumeration of real or fancied flaws in a study),
(2) Dogmatic criticism (rejection, without further analysis, of data which do not satisfy certain statistical desiderata),
(3) Speculative criticism (introduction of speculation into the critic's counter hypotheses or conclusions without establishing tenability),
(4) " Tubular " criticism (characterised by an inability to " see " evidence unfavourable to the counter hypothesis).

The examples are taken from the current controversy over the relationship between smoking and lung cancer.

(Polly Feigl)

2/498

EPSTEIN, B. (Wayne State Univ., Detroit and Stanford Univ., Cal.) **5.2** (8.6)

Statistical life test acceptance procedures—*In English*

Technometrics (1960) **2**, 435-446 (4 references, 3 tables, 2 figures)

In this paper the author gives methods for testing the hypothesis $H_0:\theta = \theta_0$ against the alternative $H_1:\theta = \theta_1$ with Type I and Type II probabilities of α and β, respectively, for a random variable which has an exponential distribution. Advantage is taken of the time-ordered nature of life test data to substantially shorten the time required to reach a decision. In the " censored tests ", n items are tested until r fail, where r is obtained from the operating characteristic or power curves. The average life of all items up to the time of the rth failure is computed and composed with a decision rule.

In the " time-truncated life-tests " a predetermined time T_0 is given. If n items are tested and r_0 fail, H_0 is rejected. If less than r fail up to time T_0, H_0 is accepted.

Finally a sequential procedure is outlined: several examples are given.

(P. H. Randolph)

2/499

JOHNSON, N. L. (University College, London) **5.7** (8.1)

On the choice of a sequential test procedure—*In English*

Symp. Quantative Methods in Pharmacology (1960)

In the construction of a standard sequential probability ratio test procedure we need to define two pivotal hypotheses H_0, H_1 and two approximate probabilities of error α_0, α_1, such that

$$\Pr\{\text{reject } H_j \mid H_j\} \cong \alpha_j, \quad j = 1, 2.$$

It is known that the sequential probability ratio test is then optimal in the sense that, among all procedures satisfying this expression, the test has nearly the smallest expected size of sample whether either H_0 or H_1 is true.

The conditions given above are the same as those often used in determining the size of sample to be taken in a classical fixed sample procedure. In such cases the hypotheses H_0 and H_1 together with the probabilities α_0 and α_1 are chosen in the light of the required sensitivity of the procedure. For example, H_0 may be a null hypothesis, and α_0 a level of significance corresponding to a desirable level of quality; with H_1 representing a change in quality of such an amount that a high probability $(1-\alpha_1)$ of detecting it is needed. If H_0 and H_1, determined in this way, are used to define a sequential probability ratio test the full advantage of the sequential method may be lost unless it is especially desirable to average minimum sample size when either H_0 or H_1 is true. When a hypothesis H, different from both H_0 and H_1, is true there is no guarantee that this test procedure defined by H_0, H_1, α_0, α_1, will even have a lower average sample size than a fixed sample procedure defined in the same way.

A sequential probability ratio test satisfying the above conditions need not necessarily be based on H_0 and H_1 as pivotal hypotheses. It is suggested that consideration should be given to the introduction of additional hypotheses H_0', H_1', to be used as pivotal hypotheses. H_0' and H_1' would be chosen so that the saving in average sample size should be as effective as possible. The associated approximate probabilities of error α_0', α_1' (where $\Pr\{\text{reject } H_j' \mid H_j' \cong \alpha_j'\}$; $j = 1, 2$) are then determined by the conditions given above.

In this paper detailed consideration is given to the case when the sequential probability ratio test is regarded as a test of the null hypothesis H_0, and where we take $H_0' \equiv H_0$. Incidentally to the main topics, some results are obtained on the comparison of operating characteristics or power functions of comparable sequential probability ratio test and ancillary tables are presented.

(N. L. Johnson)

2/500

HEMELRIJK, J. (Mathematical Centre, Amsterdam) **5.3** (5.0)

Experimental comparison of Student's and Wilcoxon's two sample tests—*In English*

Symp. Quantative Methods in Pharmacology (1960)

The choice between the many available statistical methods is often difficult to make in practical situations. It is often, too often, based on personal preference without sufficient regard for the merits of the different methods available. A lack of knowledge about the small-sample properties of many distribution-free methods tends to make their position uncertain in comparison with classical methods based on specified suppositions about the sampled populations; the properties of the latter are known, if these suppositions are true.

In a small experiment, using Quenouille's table [" Tables of random observations from standard distributions", *Biometrika* (1959) **46**, 178-204; abstracted in this journal No. **1**/325, **11.1**] fifty independent pairs of samples of size ten were used to compare student's and Wilcoxon's two-sample tests. The samples were first taken from a standard normal distribution (Quenouille's x_1), then one of them was shifted to the right over distances of 0·81; 1·05 and 1·53. These shifts correspond to a power (for Student's test) of 0·40, 0·60 and 0·90 respectively. In all these positions both tests were applied one-sidedly with level of significance 0·025. As was to be expected, Student's test being uniformly most powerful in this situation, the power of Wilcoxon's test was the smaller one.

The results indicated that in this case one would not be far off the mark in stating that, roughly, the loss of power was about ten per cent. of the power of Student's test as long as the latter is not too close to unity. This also holds under the hypothesis tested: the true level of significance for Student's test is 0·025 and 0·022 for Wilcoxon's test in this example owing to its discrete character. In about 90 or more per cent. of the cases, however, both tests led to the same conclusion about the hypothesis tested.

A second point of investigation was the reaction of the tests to an observational error in one of the original observations. For each pair of samples an observation was chosen at random from the sample with the larger median and this observation was shifted over a distance two to the right. In accordance with expectation Student's test proved to be far more sensitive to this mild kind of slippage than Wilcoxon's test. The

continued

HEMELRIJK, J. (Mathematical Centre, Amsterdam) **5.3** (5.0)

Experimental comparison of Student's and Wilcoxon's two sample tests—*In English*

continued

Symp. Quantative Methods in Pharmacology (1960)

difference between the two tests was, in this case, at least as large as the difference in reaction to the shifts mentioned in the foregoing paragraph.

Finally all observations, transformed to observations from a rather skew exponential distribution (Quenouille's x_4), were again tested in the same way. The shifts were, in this case, replaced by multiplication of the observations of one of the samples from the starting point of the distribution in order to keep this point invariant. The factors were: 1 (identical distributions); 2; 3 and 4·5, resulting in power function values of roughly 0·025, the level of significance; 0·25; 0·45 and 0·70 respectively. The two tests were now approximately equally powerful.

All these results are dependent in a statistical sense, being based on the same 1000 observations. The investigation was too small, although laborious, to warrant strong conclusions: it is only the beginning of more extensive research using more powerful means.

(J. Hemelrijk)

JUVANCZ, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **5.0** (11.0)

Contra-indication of non-parametric tests in medical experimentation—*In English*

Symp. Quantative Methods in Pharmacology (1960)

According to the author's personal experience from the design and/or analysis of over 700 experiments, most of them medical, the statisticians engaged in medical experiments show a reluctance to use non-parametric methods, though numerous theorists enthusiastically advocate them. A statistical survey is made of papers from some medical, biometrical and statistical periodicals, published in various countries. The results show the same tendency as the author has noted at home.

The reluctance of those " qualified in medicine or, if not so qualified, at least well soaked in it " is caused by some characteristic features of medical experimentation, well known to them. So, for example, in medical experiments the interest is mainly centred on the estimation of mean-values and their fiducial limits, and on material significance. There is great variation between and within individuals; the measurements are poor; the number of observations is small. Experiments are sometimes dangerous. Numerous factors disregarded in the statistical analysis influence the final decision. Experimental results are retested several times by others before acceptance and are often intended for individual use.

These characteristics compel the research workers to use statistical methods which yield results convertible to the original unit of measure. Order statistics and the χ^2-tests are not of this kind. To minimise the loss of information, non-parametric methods are too wasteful: consequently non-parametric tests are economically inefficient both qualitatively and quantitatively. Often non-parametric tests are insensitive to changes regarded as medically substantial and too sensitive to insubstantial ones. An example of this is the fact that χ^2-tests are insensitive to consistent though small increases and sensitive to great though inconsistent ones; this is not the case with the t-test. It is often impossible to decide what caused the statistically significant change indicated by a non-parametric test (mean, type of distribution, etc.). On the other hand, tests of the mean are very " robust " against non-normality. Only a small field of application is left for non-parametric methods, cases where the experimental techniques are too weak. Consequently, the use of non-parametric methods should be avoided, if possible by the invention of better experimental methods.

2/503

(I. Juvancz)

KAMENSKY, I. I. **5.3** (6.7)

On samples with two general populations and tests of hypotheses connected with them—*In Ukrainian*

Dopov. Acad. Nauk Ukrain, SSR (1960) 150-155 (4 references)

The samples $x_{11}, \ldots, x_{1r_1}$ and $x_{12}, \ldots, x_{2r_2}$ of two populations obeying the laws of distribution $\Pr(x\colon \alpha_1, \theta_1)$ $\Pr(x\colon \alpha_2, \theta_2)$ respectively are considered by the author in this paper.

The following hypotheses are examined:

(i) $H_1 - \alpha_1 = \alpha_2$; θ_1 and θ_2 are known

(ii) $H_2 - \alpha_1 = \alpha_2$; if $\theta_1 = \theta_2$ for Pareto's law.

The tests deduced are analogous to the tests obtained by Epstein & Tsao [*Ann. Math. Statist.* (1953) **24**, 458-466] for exponential distributions.

(B. V. Gnedenko)

LINHART, H. (Nat. Inst. Personnel Res., S. African Coun. Sci. Industr. Res.) **5.6** (6.5)
Approximation tests for m rankings—*In English*
Biometrika (1960) **47**, 476-480 (7 references)

One or two sets of rankings of n objects are given. The sets of size m_1 and m_2 are regarded as a random sample from a population of rankings. Approximate tests for hypotheses regarding these rankings are derived.

The first hypothesis, for a single set of markings, is that the mean ranks of a certain subset of the n are identical. The second and third hypotheses, regarding two sets of rankings are:

(i) the n mean ranks are identical in the two populations,

(ii) the coefficients of correlation are identical in the two populations.

The author uses techniques which have been previously put forward by Stuart [*Biometrika* (1951) **38**, 33-42]. He chooses statistical criteria on the basis of their reasonableness, obtains the first two moments of these, and then uses as approximations to their distributions the simplest tabulated statistical distributions: that is to say, the normal distribution for criteria within the range $-\alpha$ to $+\alpha$, and the chi-squared distribution for criteria with range zero to $+\infty$.

(Florence N. David)

2/505

MAURICE, Rita (University College, London) **5.3** (1.8)
A different loss for the choice between two populations—*In English*
J. R. Statist. Soc. B (1959) **21**, 203-213

A decision is to be made between two populations characterised by the values θ_1, θ_2 of an unknown parameter, on the basis of a sample of size n from each population.

It is argued that, for example in an industrial process producing a series of N objects, the first $2n$ of which are to be given one of two treatments and the production of the rest based on the decision between these, but, where the test pieces may also be sold, the expected loss-function should be

$$[(N-2n)p+n]\theta$$

where p is the chance of a wrong decision and $\theta = | \theta_1 - \theta_2 |$.

Reasons are given for this function to apply to a range of cases where N has the wider interpretation of " extent of use ". The differences between this and the related loss-function of Healy *et al.* and Box are discussed.

It is pointed out that the minimax principle cannot be applied (to the problem of obtaining optimal n) without further specification since the most unfavourable value of θ gives " infinite " loss; an upper bound, τ to θ is therefore supposed known. The set-up where the parameter is the mean of a normal population with known variance σ^2 is then treated in detail and it is found that there is no simple expression for optimal n; even the form of the maximigatation equation depending on τ, but numerical solutions are tabled against N and $T = \tau/\sigma\sqrt{2}$ and it is noted that unless T is as small as one or two, n_0, the minimax n does not vary much with T.

The author next considers sequential tests, $\mathscr{E}(n)$, in the minimax case, being compared with n_0. The sequential scheme is much superior on this count but it is pointed out that, owing to approximations, the comparison is only valid for large N and even then, it may not be experimentally possible to conduct sequential trials.

Finally, the expected loss for both types of scheme is plotted against θ/σ for $N = 100$, 2500 and it is seen that the sequential plan is very nearly uniformly better.

(D. E. Barton)

PEARSON, E. S. (University College, London) **5.2** (6.9)

Editorial note of Bennett and Hsu's paper " On the power function of the exact test for the 2×2 contingency table "—*In English*

Biometrika (1960) **47**, 397-398; Original paper abstracted No. **2**/469, **5.2**

Bennett & Hsu [*Biometrika* (1960) **47**, 393-397] gave a comparison of Sillito's approximation with their own computed values. The Editor of *Biometrika* here discusses Sillito's and Patnaik's approximations, pointing out that Patnaik's power values are only " exact " for the particular significance points obtained by assuming that the hypergeometric distribution under the null hypothesis could be represented by a normal distribution.

Attention is drawn to the fact that the Bennett & Hsu power diagrams bring out clearly the difficulty of distinguishing between two widely different probabilities when the samples are small.

(Florence N. David)

2/507

RAO, U. V. R., SAVAGE, I. R. & SOBEL, M. (Indiana Univ., Univ. Minnesota, Bell Telephone Labs.) **5.6** (5.3)

Contributions to the theory of rank order statistics. The two-sample censored case—*In English*

Ann. Math. Statist (1960) **31**, 415-426 (4 references)

Rank order theory is developed for the two-sample problem in which censoring of the observations has occurred. More explicitly, let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be independent random variables where the X's have distribution function F and the Y's have distribution function G. Tests of the hypothesis $F = G$ are considered. A rank order z is defined as $z = (z_1, \ldots, z_{m+n})$ where z_i is zero or one according as the ith smallest in the combined sample is an X or a Y. Rank orders may be described as paths of horizontal and vertical steps of unit length on the integer lattice from (0, 0) to (m, n) where a zero is a horizontal step, and a one is a vertical step.

In this paper a censoring scheme may be described as a set S of lattice points not containing (0, 0) such that each path from (0, 0) to (m, n) meets S. Experimentation is started at (0, 0) and continued until a point in S is reached. Under some censoring schemes the total sample size is a random variable. In the case where experimentation is continued until m^* of the X's have been observed the probability distribution of the total sample size is given for the case $F \neq G$ and for the case $F = G$. For the general type of censoring scheme described above the conditional distribution of the rank orders given the total sample size is presented in terms of F and G.

If $F(x) = H(x, 0)$ and $G(x) = H(x, \theta)$, where θ is a real parameter, a locally most powerful rank order test can be constructed by choosing a test which maximises the derivative of the likelihood ratio at $\theta = 0$. For the general censoring scheme the derivative of the likelihood ratio at $\theta = 0$ is given when $H(x, \theta)$ is the normal distribution function with mean θ, when $H(x, \theta) = 1 - [1-J(x)]^{1+\theta}$, and when $H(x, \theta) = (1-\theta)J(x)+\theta J^2(x)$ where J is a fixed distribution function. In the uncensored case these formulas reduce to known results.

In case F' and G' have a monotone likelihood ratio a partial ordering of the likelihood ratio of rank orders is given in the censored case.

(D. R. Truax)

2/508

REIMANN, J. & VINCZE, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **5.3** (5.6)

On the comparison of two samples with slightly different sizes—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1960) **5**, 293-309 (8 references)

Let $F_n(x)$ and $G_m(x)$ denote the empirical distribution functions of two samples of size n and m $(m>n)$, taken independently from populations with continuous distribution functions $F(x)$ and $G(x)$. The authors determine the distributions of the following statistics

$$B_{nm}^{+} = \max_{(x)} [nF_n(x) - mG_m(x)],$$

and

$$B_{nm} = \max_{(x)} | nF_n(x) - mG_m(x) + (m-n)/2 | - (m-n)/2$$

and the limiting distributions for the case when the sizes of the two samples only " slightly differ ", that is if $n\to\infty$ and $(m-n)^2/(m+n)\to 4c^2$, where $c>0$ is a constant. It is proved furthermore that in this case the tests based on the statistics B_{nm} and B_{nm}^{+} are asymptotically consistent against all continuous alternatives $F(x)\not\equiv G(x)$ and $F(x)>G(x)$ respectively.

The following test is suggested in such cases. (In the finite case to compensate for the lack of closer investigations $c<1$ is recommended.) Let the combination of the above-mentioned samples be denoted by $\zeta_1<\zeta_2<\ldots<\zeta_{n+m}$ arranged in order of magnitude. Let $S_i = nF_n(\zeta_i+0) - mG_m(\zeta_i+0)$ and R_{nm}^{+} and T_{nm}^{+} respectively denote the first and last of the indices i for which the sum S_i is maximal and R_{nm} the first index for which $| S_i+(m-n)/2 | -(m-n)/2$ is maximal. The authors determine the joint distributions and limiting distributions of the pairs of statistics (B_{nm}^{+}, R_{nm}^{+}), (B_{nm}^{+}, T_{nm}^{+}) and (B_{nm}, R_{nm}).

As special cases for $n = m$, that is, where $c = 0$, the distributions of Gnedenko & Korolyuk [*Dokl. Akad. Nauk SSSR* (1951) **80**, 525-528] and Vincze [*Publ. Math. Inst. Hung. Acad. Sci.* (1957) **2**, 183-209] are obtained. The authors' work is connected with the method used by Hodges in his paper [" The significance of probability of the Smirnoff two-sample test ", *Ark. Mat.* (1958) **3**, 469-486].

(K. Sarkadi)

2/509

VAART, H. R. van der (Inst. Theor. Biol., Leyden University) **5.3** (2.3)

On the robustness of Wilcoxon's two sample test—*In English*

Symp. Quantative Methods in Pharmacology (1960) (4 tables, 4 graphs)

The main object of this paper is to compare the asymptotic (for large sample sizes) behaviour of the probability of type I error for Wilcoxon's and Student's two-sample tests in the following situation: a sample of size m is drawn from a $N(\mu_1, \sigma_1^2)$ distribution, and a sample of size n is drawn from a $N(\mu_2, \sigma_2^2)$ distribution. Then, at significance level α, the null hypothesis H_0; $\mu_2 = \mu_1$ is tested against H; $\mu_2>\mu_1$, by means of the appropriate one-sided Wilcoxon's and Student's two-sample tests. The probability of rejecting H_0 when in fact it is true, depends on σ_1/σ_2 and is asymptotically equal to $Q(\lambda_\alpha/W)$ for Wilcoxon's test, and to $Q(\lambda_\alpha/S)$ for Student's test [for two-sided tests the corresponding probabilities are $2Q(\lambda_\alpha/W)$ and $2Q(\lambda_\alpha/S)$]. Here $\alpha = Q(\lambda_\alpha) = \Pr(v>\lambda_\alpha)$, v being a $N(0, 1)$ variate. Put $q = (\sigma_1^2-\sigma_2^2).(\sigma_1^2+\sigma_2^2)^{-1}$, then

$$W^2 = -3+12\pi^{-1}(m+n)^{-1}[m.\arccos(\tfrac{1}{2}(1+q)^{\frac{1}{2}}) + n.\arccos(\tfrac{1}{2}(1-q)^{\frac{1}{2}})] = 1+2.3^{\frac{1}{2}}.\pi^{-1}.q.(n-m)(n+m)^{-1} - 3^{-\frac{1}{2}}.\pi^{-1}.q^2+\ldots;$$

$$S^2 = [m+n-q.(m-n)].[m+n+q.(m-n)]^{-1} = 1 + 2q.(n-m)(n+m)^{-1} + 2q^2.(n-m)^2(n+m)^{-2}+\ldots.$$

A number of conclusions may now be drawn.

One result is for equal sample sizes, $m = n$: whereas $Q(\lambda_\alpha/S)$ is then known to equal α for any value of q (that is, for any ratio σ_1/σ_2), $Q(\lambda_\alpha/W)$ is found to vary with q and in fact to be larger than $Q(\lambda_\alpha/S)$ for each $q \neq 0$. On the other hand, if the size of one sample is $k(\geqq 2)$ times the size of the other sample, Wilcoxon's $Q(\lambda_\alpha/W)$ deviates from α less than does Student's $Q(\lambda_\alpha/S)$, again for each $q \neq 0$. Finally, if the size of one sample is $k(1<k<2)$ times the size of the other sample, Wilcoxon's test behaves better only on part of the q-range. In the immediate neighbourhood of $q = 0$, that is, of $\sigma_1 = \sigma_2$, the ratio $Q(\lambda_\alpha/W)/Q(\lambda_\alpha/S)\approx 3^{\frac{1}{2}}.\pi^{-1}\approx 0{\cdot}55$. Four tables are given for numerical values of $Q(\lambda_\alpha/W)$ and $Q(\lambda_\alpha/S)$ for $n = m$, $n = \frac{3}{2}m$, $n = 2m$, $n = 3m$, $(\alpha = 0{\cdot}05)$ together with four graphs.

Summarising, in the two-sample problem in the case of two normal distributions with unequal variances, the asymptotic behaviour of the probability of type I error for Wilcoxon's test is not uniformly better than for Student's test; but Wilcoxon's test behaves rather worse only for rather extreme values of σ_1/σ_2, combined with certain values of the ratio m/n between $\frac{1}{2}$ and 2.

(H. R. van der Vaart)

2/510

VINCZE, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **5.3** (5.6)
On a special type of randomised tests—*In English*
Second Hungarian Mathematical Congress (1961)

Let $t_{nm} = t_{nm}(\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \ldots, \eta_m)$ be a statistic used for deciding whether the null-hypothesis $F(x) \equiv G(x)$ is true. Here $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_m$ are samples taken from populations with the continuous distributions $F(x)$ and $G(x)$ respectively. Let us consider the case when t_{nm} is not asymptotically consistent against all continuous alternatives.

The author shows some examples in which t_{nm} may be imbedded in a parametric set of statistics

$$\tau_{nm}(\vartheta) = \tau_{nm}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m;\ \vartheta)\vartheta \in \Omega,\ \tau_{nm}(\vartheta_0) = t_{nm}$$

such that (i) in case the null-hypothesis holds the distribution of $\tau_{nm}(\vartheta)$ does not depend on ϑ, (ii) choosing ϑ randomly, the randomised test obtained will be with probability one (relative to the measure defined on Ω) asymptotically consistent against all continuous alternatives; that is to say, for any given $G(x) \not\equiv F(x)$ with probability zero will be chosen so that the test based on $\tau_{nm}(\vartheta)$ will not be consistent.

(I. Vincze)

2/511

WEISS, L. (Cornell University, Ithaca, New York) **5.2** (2.6)
A test of fit based on the largest sample spacing—*In English*
J. Soc. Indust. Appl. Math. (1960) **8**, 295-299 (3 references)

A test-of-fit is presented based on V, the largest of the $n+1$ sample spacings obtained when the probability transformation is made on n independent, identically distributed random variables $(x_1, x_2, \ldots, x_n)$ from a continuous unknown distribution. The completely specified function $F(x)$ used for the transformation is the hypothesised cumulative distribution function.

The ordered values on the interval [0, 1] are denoted $Y_1 \leqq Y_2 \leqq \ldots \leqq Y_n$. Y_0 denotes 0 and Y_{n+1} denotes 1. T_i denotes $Y_i - Y_{i-1}$ for $i = 1, \ldots, n+1$. $T_1, T_2, \ldots, T_{n+1}$ are the $n+1$ sample spacings of which V is the largest.

When the hypothesis is true the distribution of V is:

$$\Pr(V<d) = \sum_{0 \leqq j < \min\left(\frac{1}{d},\ n+2\right)} \binom{n+1}{j} (-1)^j (1-dj)^n$$

and

$$\lim_{n\to\infty} \Pr\left[V < \frac{\log(n+1)+a}{n+1}\right] = \exp(-e^{-a})$$

The test rejects when $V \geqq d_n(\alpha)$. The critical value $d_n(\alpha)$ depends on the sample size, and the desired level of significance α.

Properties of the test were investigated by the author and it was shown to be admissible and consistent.

(D. T. Searls)

ARMITAGE, P. & BARTSCH, G. E. (London School of Hygiene and Tropical Medicine) **6.8** (5.6)
The detection of host variability in a dilution series with single observations—*In English*
Biometrics (1960) **16**, 582-592 (13 references, 4 tables, 1 figure)

Current techniques are investigated for testing the hypothesis of equal host susceptibility to infection from a single particle of an inoculum. Attention is restricted in the paper to the case of single observations that form a two-fold dilution series. The Stevens range statistic R, the number of dilutions between (and including) the first at which not all observations are positive, that is to say the infection observed, and the last at which not all are negative is compared with a new statistic J, the number of positive results at dilutions beyond that at which the first negative occurs. It is shown that the asymptotically efficient statistics ϕ and ϕ' due to Armitage are more closely related to J than to R. The distribution of J under the null hypothesis of equal susceptibility is computed for a two-fold dilution series. The power function for the J test is calculated with host variability a binomial (half of the hosts with susceptibility level p_0, half with level $1-p_0$) variable. The J-statistic appears to have a steeper power curve for small departures from the null hypothesis, while for large values of the departure parameter $\log[(1-p_0)/p_0]$ this advantage seems to disappear.

(D. F. Morrison)

2/513

BARTEN, A. P. (Central Planning Bureau, The Hague, Netherlands) **6.1** (6.0)
Residual terms—a source of ordered information in regression analysis—*In Dutch*
Statist. Neerlandica (1960) **14**, 319-341 (24 references, 6 figures)

Although in regression analysis residual terms seem to represent chaos, they can be used to complement the original hypothesis in a theoretically acceptable way. This is demonstrated with an example taken from the construction of a wage-rate equation for the Dutch economy 1923-38 and 1949-57. Extrapolation of the equation for the years 1958 and 1959 and its application to the United States economy 1931-58 provide a primitive test.

(A. P. Barten)

2/514

BERKSON, J. (Mayo Clinic, Minnesota) **6.7** (11.1)

Nomograms for fitting the logistic function by maximum likelihood—*In English*

Biometrika (1960) **47**, 121-141 (8 references, 4 tables, 8 nomograms)

In this paper the author uses the logistic function as a model for bio-assay and other experiments where a quantal response is observed at different values of a stimulus. In such experiments, as well as estimating the two parameters, α and β of the function, it is frequently desired to estimate the L.D.$_{50,\,\alpha}$ given by their ratio.

If r animals out of n die when they are given dose x then the maximum-likelihood estimates of α, β and γ are functions of the two quantities; the sum of r over all values of x and the sum of the product rx over all the values of x. It was pointed out that, for any specified dosage arrangement, maximum-likelihood estimates could be tabled against an argument of these two quantities. The author points out that for the required accuracy the tabular presentation would be formidably elaborate and the computation would be quite tedious: he has therefore preferred nomographic presentation.

The nomograms presented provide maximum-likelihood estimates of γ and β for equally spaced dosages and equal n at each dose, for 3, 4, 5 and 6 doses. A separate nomogram is given for γ and for β.

The variances of the estimates, obtained from large sample formulae, are determined by n and the estimates of γ and β. The author gives in eight pages of tables the standard error of each estimate, multiplied by $\sqrt{n}$, to two significant figures.

Two examples of the use of the nomograms and tables are given.

(N. W. Please)

2/515

BLOEMENA, A. R. (Mathematical Centre, Amsterdam) **6.9** (2.9)

Random associations of points on a graph—*In Dutch*

Statist. Neerlandica (1960) **14**, 267-274 (14 references, 1 figure)

The author supposes a graph consisting of n points, numbered from 1, ..., n, and m_{ij} joins between points i and j. From this graph r points are chosen at random. This paper deals with the properties of a random variable based upon the number of joins between chosen points. Applications to a test for randomness, and to the order-disorder problem are considered.

Analogous results for other statistics and also for the case in which the points are sampled independently, are given. These results are described in Bloemena's report: " On probability distributions arising from points on a graph " [*Statist. Dept., Math. Centre*, Report S 266A (1960); abstracted in this journal No. **1**/540, **1.3**].

(A. R. Bloemena)

2/516

CSÁKI, P. & FISCHER, J. (Math. Inst., Hungarian Academy of Sciences, Budapest) **6.2** (6.9)
On bivariate stochastic connection—*In English*
Second Hungarian Mathematical Congress (1961)

In this paper the problems of bivariate connection are discussed with the aid of Hilbert space theory. Let $L^2 = L^2(\Omega, S, P)$ be the Hilbert space of random variables of finite variance defined on the probability space (Ω, S, P). Let ξ and η be arbitrary random variables with joint distribution function $H(x, y)$ and marginal distributions $H_1(x)$ and $H_2(y)$, respectively, which generate the measure P on the plane (x, y) and the measures P_1 and P_2 on the real line. Let $L^2_{H_i}$ $(i = 1, 2)$ denote the space of functions $f(x)$ for which $\int_{-\infty}^{\infty} f^2(x)dH_i(x)$ is finite; and $L^2_\xi = L(\Omega, S_\xi, P)$ where S_ξ denotes the smallest σ-algebra with respect to which ξ is measurable; L^2_η is defined analogously. $L^2_{H_1}$ and L^2_ξ are isomorphic. The sub-space of L^2_ξ consisting of its elements with zero expected values will be denoted by $L^2_{\xi,0}$: $L^2_{\eta,0}$ is defined analogously.

Let A_η and A_ξ be the operators transforming the elements of L^2 into their orthogonal projection on L^2_η and L^2_ξ respectively. Thus the conditional expected value of $\zeta \in L^2$ on η is $M(\zeta/\eta) = A_\eta\zeta$. The maximal correlation and the mean-square contingency of ξ and η are $S(\xi, \eta) = \| A_\eta \| = \| A_\xi \|$ and $C(\xi, \eta) = ||| A_\eta ||| = ||| A_\xi |||$, whenever the domains of A_η and A_ξ are restricted to $L^2_{\xi,0}$ and $L^2_{\eta,0}$ respectively.

The operator A defined on $L^2_{H_1}$ by the isomorphism of L^2_ξ and $L^2_{H_1}$ which corresponds to A_η/A_2 is defined analogously on $L^2_{H_2}$.

The equivalency of the following statements is proved:

(i) $P_1(A \mid y) \ll P_1(A)$ and $P_2(B \mid x) \ll P_2(B)$

(ii) $P \ll P_1 \times P_2$

(iii) A_1 and A_2 are integral operators, where $P_1(A \mid y)$ and $P_2(B \mid x)$ are the conditional distributions, A and B are Borel sets on the real line: this is Theorem 1.

It is proved that the above definition of the mean-square contingency is equivalent to that of Rényi [New version of the probabilistic generalisation of the large sieve, *Acta Math. Acad. Sci. Hung.* (1959) **10**, 217-226; abstracted in this journal No. **1**/252, **6.0**; and On the measure of dependence, *Acta Math. Acad. Sci. Hung.* (1959) **10**, 441-451; abstracted in this journal No. **1**/658, **6.0**.

Theorem two provides a method for replacing any given distribution on the plane with maximal correlation S by a symmetric one with maximal correlation S^2.

(P. Csáki)

2/517

FISCHER, J. & CSÁKI, P. (Math. Inst., Hungarian Academy of Sciences, Budapest) **6.2** (6.0)
Contributions to the problem of maximal correlation—*In English*
Second Hungarian Mathematical Congress (1961)

Let ξ and η be arbitrary random variables defined on the probability space (Ω, S, P) and S_ξ denote the smallest σ-algebra with respect to which ξ is measurable, further $L^2_\xi = L^2(\Omega, S_\xi, P)$. The subspace of L^2_ξ consisting of its elements with zero expected values will be denoted by $L^2_{\xi,0}$. The σ-algebra S_η, the spaces L^2_η and $L^2_{\eta,0}$ are analogously defined. The symbol A_η denotes the operator of the orthogonal projection of the elements of $L^2_{\xi,0}$ on $L^2_{\eta,0}$; the conditional expected value, and A_ξ is defined analogously.

The correlation ratio of a standard random variable $f \in L^2_{\xi,0}$ on η is $\theta_\eta(f) = \| A_\eta f \|$, the maximal correlation of ξ and η is $S(\xi; \eta) = \| A_\eta \| = \| A_\xi \|$.

If $A_\eta f = \lambda g$ and $A_\xi g = \lambda f$ holds then we call λ an eigenvalue and f, g a pair of eigenfunctions of the pair of operators A_η, A_ξ. Necessary and sufficient conditions are given for $f \in L^2_{\xi,0}$ and $g \in L^2_{\eta,0}$ to form a pair of eigenfunctions; theorem one.

In the following the case $S(\xi, \eta) = 1$ is classified and a necessary and sufficient condition for $\eta = f(\xi)$ is given.

In the case of linearly correlated random variables ξ and η a necessary and sufficient condition for homoscedastical correlation is described in theorem two. Further, theorem three provides a method for calculating the eigenvalues and eigenfunctions of the pair of operators A_η, A_ξ whenever linearly independent systems $\phi_n \in L^2_{\xi,0}$; $\psi_n \in L^2_{\eta,0}$ satisfying certain conditions are known. The eigenfunctions are polynomials if and only if for each n the nth conditional moment is at most an n-degree polynomial of the conditioning variable.

Finally, examples are presented for calculating the maximal correlation.

(J. Fischer)

2/518

FRÉCHÉT, M. (University of Paris)
Tables which have given margins—*In French*
Trab. Estadíst. (1960) **9**, 3-18

6.9 (6.2)

The author considers a rectangular table of q rows and r columns; the element of the j row, i column will be designed by $n_{ij} \geqq 0$.

The margins of this table are the sums:

$$N_i = \sum_{j=1}^{r} n_{ij};\ N'_{ij} = \sum_{i=1}^{q} \text{ where } \sum_i N_i = \sum_j N'_j.$$

The following problem is considered: what may be said about a table when the margins are known? More precisely, when given the non-negative numbers $N_1, N_2, \ldots, N_q$; $N'_1, \ldots, N'_r$ we have to find

(i) If there exists a table such that the given numbers are their margins.
(ii) If there are one or more.
(iii) How to calculate the elements of a table which is a solution of the problem; supposing that this solution exists.

The author proves that there exists at least one table which has the given numbers as margins. He finds an expression which allows him to obtain all the possible solutions; special care is given to the extremal solutions.

As a generalisation, the distribution functions are considered: if $F(x)$ is the frequency in which $X<x$ and $G(y)$ is the frequency in which $Y<y$ we have to determine a distribution function $H(x, y)$ such that $F(x)$ and $G(y)$ are the marginal distributions. It is proved that the necessary and sufficient condition for this is that $H(x, y)$ be a function included between $H^0(x, y)$ and $H^1(x, y)$, where

$$H^0(x, y) = \mathrm{Max} \begin{matrix} 0 \\ F(x)+G(y)-1 \end{matrix}$$

$$H^1(x, y) = \mathrm{Min} \begin{matrix} F(x) \\ G(y) \end{matrix}$$

The following two applications to linear programmes are considered:

(i) when the frequency laws of X and Y are given, the minimum and the maximum of the former coefficient of correlation of X and Y is the coefficient given by H^0 and H^1, that is, is given by the extremal solutions to the right and to the left

2/519

continued

FRÉCHÉT, M. (University of Paris)
Tables which have given margins—*In French*
Trab. Estadíst. (1960) **9**, 3-18

6.9 (6.2)
continued

(ii) another application of these properties is to give a definition to the distance between two statistical distributions $F(x)$ and $G(x)$.

In the second part of the paper the author considers analogous problems, by regarding X and Y as simultaneously determined random variables and examines the properties of the correlation between X and Y when the probability laws of these variables are known. It is proved that there are at least two distinct solutions, $H^0(x, y)$ and $H^1(x, y)$, which are respectively the greater and the smaller of the solutions. The author also studies the case of a unique solution and he proves that the necessary and sufficient condition for the existence of such solution is that one at least of the random numbers x, y is a constant. In this case the unique solution $H(x, y)$ is identical to $H^0(x, y)$ and to $H^1(x, y)$.

(J. Béjar)

2/520

HEY, E. N. & HEY, M. H. (Radcliffe Infirmary, Oxford and British Museum, London) **6.7** (4.8)

The statistical estimation of a rectangular hyperbola—*In English*

Biometrics (1960) **16**, 606-617 (3 references, 3 tables, 2 figures)

A simple method is described for fitting a function of the type $(X-a)(Y-b) = c$ to experimental data in two variables x and y, both subject to errors of measurement. The errors are assumed to be normally and independently distributed but in general the error variances may be different, not only for the two variables but also for different observations of the same variable. Maximum likelihood estimates of a, b and c are obtained by minimising the sum of the squares of standardised residuals in both variables. A direct and rigorous solution of this minimal problem is possible only in a very few special cases, the most important of which is when a straight line is to be fitted to the data and the ratio of the error variances is constant for all points.

Brown [*J. Pharmacol.* (1952) **105**, 139-155] fits a rectangular hyperbola to experimental data by minimising the sum of the squares of the differences between the area of the rectangle defined by the asymptotes of the estimated curve and an experimental point and the rectangle of constant area c defined by the asymptotes and any point on the estimated curve. The solution is not entirely satisfactory; the method gives excessive weight to points with high values of x or y.

The present authors estimate the parameters a, b and c by minimising the sum of the squares of standardised " normal " residuals, that is, distances from the experimental points at right angles to the fitted curve. Two approximations are introduced. First, the quantity actually minimised is an approximation to the sum of squared normal residuals. Secondly, estimates of the parameters are required for the minimisation. These can be obtained graphically from the data. Improved estimates can be obtained by iteration but the authors have not found this necessary in practice as the solutions are not very sensitive to variations in the first estimates of the parameters.

The computation procedure is illustrated by a numerical example. The estimated curve fits the data better than that obtained using Brown's method. The present procedure is well adapted for either a desk calculator or an automatic electronic machine. Formulæ for the approximate variances of the estimates are also given; the paper concludes with an appendix giving the expectation of the best estimate of the root-mean-square standardised normal residual, used for testing the goodness-of-fit of the estimated curve.

2/521 (V. Chew)

HUDIMOTO, H. (Institute of Statistical Mathematics, Tokyo) **6.5** (3.8)

On a coefficient of unidimensional ordering for the individuals' attitudes—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 27-35 (4 references, 4 figures)

Suppose that responses for m questions $(\Pi_1, \ldots, \Pi_m)$ which characterise a sociological or a psychological object A in some sense are available for the purpose of grading of individuals in a certain group according to the intensity of attitudes for A.

In this paper the author considers the dichotomous case that an individual's response to every Π_i can be reckoned to take a score of unity or zero according to favour or disfavour; the grading is carried out by their total scores. The adequacy of such a procedure is discussed, and the comparison of the set of individuals' response patterns which actually occurred with the ideal set of individuals' response patterns (denoting that the procedure is completely adequate) is given by the ratio of variances calculated from the total scores. It is shown that the value of this ratio is non-negative, and does not exceed unity. For the cases where the responses to different items are statistically independent, it is shown that the ratio of covariances is approximately distributed according to a normal distribution for a large number of respondent individuals.

(H. Hudimoto)

2/522

LANCASTER, H. O. (University of Sydney) **6.2** (2.1)

On statistical independence and zero correlation in several dimensions—*In English*

J. Aust. Math. Soc. (1960) **1**, 492-496 (4 references)

The author has extended his earlier theorem on the independence of two random variables [*Aust. J. Statist.* (1959) **1**, 53-56; abstracted in this journal No. **1**/435, **6.2**] to an arbitrary number of dimensions. The following results are obtained:

Theorem 1. Let $\{x_1^{(i_1)}\}$ and $\{x_2^{(i_2)}\}$ be complete orthonormal sets on the marginal distributions of two random variables, which have a joint probability measure $P \equiv P(x_1, x_2)$, and let $x_1^{(0)} = x_2^{(0)} = 1$. Then a necessary and sufficient condition for the independence of x_1 and x_2 is that every coefficient of correlation

$$\rho_{i_1 i_2} = \int x_1^{(i_1)} x_2^{(i_2)} dP,$$

should be zero for every $i_1 > 0$, $i_2 > 0$.

Theorem 2. The random variables $\{x_j\}$ are mutually independent if and only if the generalised coefficients of correlation $\rho^{\{i_j\}}$, corresponding to complete sets of orthonormal functions, are all zero.

(J. Gani)

2/523

NADDEO, A. (Faculty of Statistics, University of Rome) **6.9** (11.0)

On the measure of statistical dependency between two disjoint qualitative variables—*In Italian*

Riv. Ital. Econ. Demogr. Statist. (1960) **14**, 117-130 (3 references, 6 tables)

The author, making use of the law of conservation of formal properties, establishes first an index of dissimilarity between two disjoint qualitative variables. Using this result he extends them to the joint distribution of the two variables the indices of connection by Gini & Castellano; the index of similarity, also by Gini, and the Pearson's correlation ratio, and shows in some cases a few simple relations connecting these indices to the chi-squared distribution.

(V. Levis)

2/524

PERKAL, J. & SZCZOTKA, F. (Math. Inst., Polish Academy of Sciences, Wroclaw) **6.4** (6.3)
A new method for analysis of multiple attributes—*In German*
Biom. Zeit. (1960) **2**, 108-116 (3 references, 8 tables)

The analysis of multiple characters in classification and allied multivariate problems of a mainly descriptive nature by factor-analysis methods is cumbersome and frequently prone to interpretational fallacies. This paper is concerned with a simpler and more straightforward method of extracting a reduced set of " characteristic indices " from a set of p correlated attributes; sampled on each of n individuals and measured in a standardised scale. The np sample variates x_{ij}, $i = 1, \ldots, p$; $j = 1, \ldots, n$ constitute a $p \times n$ matrix $\boldsymbol{x}$ of p row vectors of elements x_i with zero mean and unit mean-square and n column vectors x_j with means m_j. In case (i) of a positive correlation matrix $\boldsymbol{R}_0 = (1/n)(\boldsymbol{xx}')$ a single " first order index " q_j^I of the jth individual is uniquely determined and equal to the individual mean m_j, $q_j^I = m_j$. The standardised first residuals $d_{ij} = (x_{ij} - q_j^I)(1/s_i)$, where $ns_i^2 = [x_i - q^I][x_i - q^I]^1$, form a $p \times n$ matrix $\boldsymbol{D}_1$ of p unit row vectors d_i. The attributes are renumbered and arranged in s groups of p_k row vectors $d_i^{(k)}$ in such a way that the s sub-matrices $\boldsymbol{D}_{1k}$ ($k = 1, \ldots, s$) have positive correlation matrices $\boldsymbol{R}_{1k} = (1/n)(\boldsymbol{D}_{1k}\boldsymbol{D}'_{1k})$, where $\Sigma p_k \leq p$.

Certain rules are given to avoid ambiguities in grouping of the d_i. The column means $\bar{d}_j^{(k)}$ of the matrices $\boldsymbol{D}_{1k}$ constitute s sets of second order indices $m_j^{II,k}$. Higher order indices can be obtained in the same way as long as the residual correlation matrices contain positive sub-matrices. If the correlation matrix $\boldsymbol{R}_0$ is non-positive the matrix $\boldsymbol{x}$ is rearranged by rows and partitioned in such a way that the sub-matrices have positive correlation matrices.

In a first example ($p = 7$ human body measurements, $n = 195$ male individuals) the data are fitted by extracting one first and two second order indices; in a second example comparisons with factor methods show that this method has a low relative error.

(R. Wette)

2/525

TALLIS, G. M. (Commonw. Sci. Industr. Res. Org., Glebe, New South Wales) **6.4** (4.1)
The sampling errors of estimated genetic regression coefficients and the errors of predicted genetic gains—*In English*
Aust. J. Statist. (1960) **2**, 66-77 (12 references, 3 tables)

The purpose of this paper is to develop a theory, general yet easy to put into practice, for the evaluation of sampling variances of estimated genetic regression coefficients, and genetic gains.

Two main methods are known for estimating genetic regression coefficients. Method (*a*) uses relationships between parents and offspring, and is equivalent to an ordinary multiple regression analysis. Method (*b*) depends on relationships between full-sibs or half-sibs.

Sampling errors are obtained for genetic regression coefficients estimated by methods (*a*) and (*b*). Sampling errors of predicted genetic gains are also found for method (*a*); exact results cannot be derived for method (*b*), but an approximation is given.

Some special cases of particular interest are considered, and an extension of the method to the estimation of the mean of the progeny from selected parents given. The paper concludes with an illustrative example based on some wool data.

(J. Gani)

2/526

CALIŃSKY, T. (Inst. Plant Breeding, Polish Academy of Sciences, Poznan) **7.5** (7.2)

On a certain statistical method for investigating interaction in several experiments with plant varieties—*In English*

Second Hungarian Mathematical Congress (1961)

Put

$$x_{ij} = m+p_i+v_j+w_{ij}+\varepsilon_{ij}\,(i = 1, 2, ..., n;\ j = 1, 2, ..., k)$$

where m, v_j are constants, p_i, w_{ij} and ε_{ij} are mutually independent normal random variables, each with zero mean and with variances σ_p^2, $\sigma_{w_j}^2$, σ_ε^2/v respectively.

The paper deals with the statistical investigation of the interaction variances $\sigma_{w_1}^2, \sigma_{w_2}^2, ..., \sigma_{w_k}^2$.

Put

$$s_j^2 = \frac{k}{(n-1)(k-1)} \sum_{i=1}^{n}\left(x_{ij}-\frac{1}{k}\sum_{r=1}^{k} x_{ir}-\frac{1}{n}\sum_{t=1}^{n} x_{tj}+\frac{1}{nk}\sum_{t=1}^{n}\sum_{r=1}^{k} x_{tr}\right)^2$$

and let s_0^2 denote the pooled experimental error. In this paper the author develops firstly the effective test for significance of particular components of interaction, based on the " mean-square ratio " s_j^2/s_0^2, and secondly, the approximate test for heterogeneity of interaction, based on the ratio of one s_j^2 to another; for example s_2^2/s_1^2.

In both the tests the F distribution is used. While the development of the former test is comparatively simple, the latter requires a more complicated mathematical treatment. Among others, the non-central χ^2 distribution and the negative binomial distribution are applied.

The practical application of the method is demonstrated on the analysis of the results of a series of experiments with varieties of garden pea.

(K. Sarkadi)

2/527

DANFORD, M. B., HUGHES, H. M. & McNEE, R. C. (School of Avn. Med., Brooks Air Force Base, Texas) **7.5** (5.8)

On the analysis of repeated-measurements experiments—*In English*

Biometrics (1960) **16**, 547-565 (12 references, 6 tables, 3 figures)

Suppose there are I levels of some treatment, with n_i subjects exposed to the ith level, and each subject measured on some characteristic periodically for K times after treatment. The problems considered are those of obtaining tests of hypotheses about treatment, time, and treatment by time interaction effects. Difficulties are encountered with performing a univariate analysis of variance because the assumption of equal variances and covariances is not fulfilled.

Univariate procedures are discussed, with the application of the techniques to the data of a particular example. Multivariate techniques are considered, together with their underlying assumptions. The latter are applied to the data of the example, and a comparison of the univariate and multivariate tests is made. A serial correlation model is also considered. It is noted that asymptotically the univariate and multivariate tests are identical. The conclusions are that, for the example given, essentially the same inferences are made from the univariate and multivariate analyses.

(W. A. Glenn)

2/528

DEDOLPH, R. R. (University of Maryland) **7.5** (6.1)

A suggested method for handling data obtained with an exponential (variable dosage) sprayer—*In English*

Proc. Amer. Soc. Hort. Sci. (1960) **75**, 789-798 (6 references, 5 tables, 4 figures)

With the development of the exponential or variable dosage sprayer, described in detail by Dedolph, Basham & Stark [*Proc. Amer. Soc. Hort. Sci.* (1960) **75**, 785-788], there has arisen a need for a method of handling data taken from experiments utilising this machine. The sprayer, designed to facilitate concentration screening of growth regulators, herbicides, fungicides, etc., operates on the principle of continual dilution, and applies the affecting chemical at a rate which decreases exponentially with row distance covered. Randomisation of concentration is therefore impossible, although the direction of spray runs, that is, replications, can be randomised.

In order to apply the analysis of variance to data of this type, one must be willing to assume that the correlation between adjacent plots, that is to say, between adjacent row increments, is negligible. Using yield per row increment, or per segment of a row increment, as the dependent variable and distance in increments as the independent variable, the standard curvilinear regression procedures for fitting the response curve and for estimating confidence limits are applicable. Since row distance covered and concentration of chemical are functionally related, the response curve for yield can be expressed in terms of the concentration. The significance of the confidence limits in determining the initial concentration of chemical to use is discussed.

(T. R. Konsler)

2/529

HARTER, H. L. (Aero. Res. Labs., Wright-Patterson Air Force Base, Ohio) **7.7** (11.1)

Critical values for Duncan's new multiple range test—*In English*

Biometrics (1960) **16**, 671-685 (8 references, 1 table)

The purpose of this paper is to give revised, extended and additional tables of critical values for D. B. Duncan's well-known multiple-range test [see, for example, *Biometrics* (1955) **11**, 1-42 and (1957) **13**, 164-176]. The table entries are upper 100 per cent. points of the studentised range for a normal parent population, with parameters p, the number of variates whose range is being taken and v, the degrees of freedom on which the estimate of error is based. In turn, the "protection level" P equals $(1-\alpha)^{p-1}$, where α is the significance level. Entries are to four significant figures for the following values of the parameters: $p = 2\ (1)\ 20\ (2)\ 40\ (10)\ 100$, $v = 1\ (1)\ 20, 24, 30, 40, 60, 120, \infty$ and $\alpha = 0{\cdot}10, 0{\cdot}05, 0{\cdot}01, 0{\cdot}005, 0{\cdot}001$.

(H. A. David)

2/530

LE ROY, H. L. & LANDIS, J. (Swiss Federal Institute of Technology, Zurich) **7.4** (-.-)

The estimation of missing values in $a \cdot b \cdot c$- and $a \cdot b$-experiments—*In German*

Biom. Zeit. (1960) **2**, 98-107 (3 references, 3 tables)

Unbiased estimators $\hat{x}_{ijk}$ of n missing values x_{ijk} in $a \cdot b \cdot c$-experiments are obtained by minimising the abc-interaction mean-square. The estimation procedure reduces to the solution of the system of equations $\hat{x} = K^{-1}s$, where $\hat{x}$ is the vector of estimates, s a vector with arbitrarily ordered elements

$$S_{ijk} = abT_{ij.} + acT_{i.k} + bcT_{.jk} - aT_{i..} - bT_{.j.} - cT_{..k} + T_{...},$$

where the T_{ijk} etc. are the observed sub-totals ($i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, c$), and K^{-1} is the inverse of a $n \times n$ matrix K, the elements of which are easily determined in the following way:

(i) put the n missing values in the same arbitrarily chosen (preferably lexicographical) order as the S_{ijk} according to index ijk, let this order determine rows ($r = 1, \ldots, n$) and columns ($s = 1, \ldots, n$) of K, and write down a marginal row and column of this indices.

(ii) let $\alpha = 1-a$, $\beta = 1-b$ and $\gamma = 1-c$ and determine the elements of K according to

$$k_{rs} = -(\alpha^{\delta_{ii'}} \cdot \beta^{\delta_{jj'}} \cdot \gamma^{\delta_{kk'}}),$$

where $\delta_{ll'}$ is the kronecker-delta, that is according to concordance ($i = i'$ etc.) or discordance of the marginal row and column indices ijk and $i'j'k'$ respectively. The solution of the system of equations then follows along the usual lines.

In case of an $a \cdot b$-experiment simply put

$$S_{ij} = aT_{i.} + bT_{.j} - T_{..}$$

and

$$k_{rs} = \alpha^{\delta_{ii'}} \cdot \beta^{\delta_{jj'}}$$

and proceed as before. Two numerical examples are given.

(R. Wette)

2/531

MORLEY, F. H. W. (Commonw. Sci. Industr. Res. Organisation, Canberra) **7.2** (9.1)

An effect of genotype $\times$ environment interactions on estimation of genotypic covariances—*In English*

Aust. J. Statist. (1960) **2**, 97-101 (4 references, 3 tables)

Kempthorne [*Introduction to Genetic Statistics* (1957) New York: Wiley] has discussed the estimation of components of genotypic variance in a model in which s sires were each mated to a random sample of d dams, the progeny of each cross being tested in r replicates of a randomised block experiment.

Certain assumptions are made in this model which the author has found inapplicable to his own recent studies in alfalfa. A modified model is therefore introduced, its main difference being that the number of replicates may be random if necessary. In this way, unbiased estimates can be more readily obtained even if interactions are present, providing these are included in the new model. Examples of the analysis when replicates are random and fixed are given.

(J. Gani)

2/532

RASCH, D. (Inst. Tierzuchtf. Landwirt.-wissensch., Berlin) **7.5** (7.0)

On the analysis of variance for two-way classification with unequal numbers in subclasses—*In German*

Biom. Zeit. (1960) **2**, 194-203 (8 references, 4 tables)

After an explanation of the orthogonal case of the analysis of variance technique for two-way classification and an illustration of the fact that the additive property of the sums-of-squares is still valid if the numbers in subclasses are proportionate, the author presents three approximate methods for the non-orthogonal case of analysis of variance.

(i) Snedecor's "method of unweighted means" [*Statistical Methods* (1946) Ames: Iowa State Press, pp. 285-301] where the marginal means are estimated by $x'_{i\cdot\cdot} = 1/q \sum_j x_{ij\cdot}$, or $x'_{\cdot j\cdot} = 1/p \sum_i x_{ij\cdot}$, respectively, and the total mean-estimate is

$$x'_{\cdot\cdot\cdot} = 1/pq \sum_{i,j} x_{ij\cdot},$$

and the error-sum of squares is given by the intra-subclass sum-of-squares divided by $pq/\sum_{i,j} \frac{1}{n_{ij}}$;

(ii) Kendall's "method of weighted squares of means" [*Advanced Theory of Statistics* (1946) **2**, London: Griffin, pp. 220-228] where the variance between the $\bar{x}_{i\cdot\cdot}$ (and the $\bar{x}_{\cdot j\cdot}$) is calculated using weights $g'_{i\cdot} = q^2/\sum_j \frac{1}{n_{ij}}$, or $g'_{\cdot j} = p^2/\sum_i \frac{1}{n_{ij}}$, and the error sum-of-squares is given by the bare intra-subclass sum of squares;

(iii) Cramér's "method of weighted squares of means" [*Mathematical Methods of Statistics* (1946) Princeton, pp. 536-547] where the marginal means are estimated by

$$x''_{i\cdot\cdot} = 1/n_{\cdot\cdot} \sum_j n_{\cdot j} x_{ij\cdot}, \text{ or } x''_{\cdot j\cdot} = 1/n_{\cdot\cdot} \sum_i n_{i\cdot} x_{ij\cdot},$$

respectively, their squares are weighted by

$$g''_{i\cdot} = n^2_{\cdot\cdot}/\sum_j \frac{n^2_{\cdot j}}{n_{ij}}, \text{ or } g''_{\cdot j} = n^2_{\cdot\cdot}/\sum_i \frac{n^2_{i\cdot}}{n_{ij}},$$

and the error sum of squares is given by the bare intra-subclass sum of squares.

Annotations: x_{ijk} is the kth variable in subclass (i, j), $k = 1, 2, \ldots, n_{ij}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Pointed

continued

RASCH, D. (Inst. Tierzuchtf. Landwirt.-wissensch., Berlin) **7.5** (7.0) *continued*

On the analysis of variance for two-way classification with unequal numbers in subclasses—*In German*

Biom. Zeit. (1960) **2**, 194-203 (8 references, 4 tables)

subscripts are written for mean operations in x's or summations in n's. Marked letters stand for estimates or weights.

It is stated that Snedecor's mean estimates are not as efficient as Cramér's, and that Cramér's method is more powerful than Kendall's, if

$$\sum_i g''_{i\cdot} - \sum_i (g''_{i\cdot})^2/\sum_i g''_{i\cdot} > \sum_i g'_{i\cdot} - \sum_i (g'_{i\cdot})^2/\sum_i g'_{i\cdot}$$

and if an analogous inequality simultaneously holds for j. The power of the different methods is illustrated, by a numerical example with x the run of food of pigs (until to a living weight of 110 kg.), i the sex of pigs, and j several years.

(R. K. Bauer)

ROBSON, D. S. & ATKINSON, G. F. (Cornell University, Ithaca, New York) **7.6** (6.1)
Individual degrees-of-freedom for testing homogeneity of regression coefficients in a one-way analysis of covariance—*In English*
Biometrics (1960) **16**, 593-605 (5 references, 4 tables, 1 figure)

The test procedure for the one-way covariance design presented in this paper is constructed specifically to have power against some restricted classes of heterogeneous but still linear alternatives to the hypothesis of homogeneous (parallel) linear regressions. If the regression lines do not intersect at a common point, which is an imposed regularity condition, then the relationship between slope and intercept will be non-linear. Whatever the configuration of t regression lines, the slopes may be expressed as a polynomial function of the intercepts, of degree at most $(t-1)$. If the within-treatment slopes are homogeneous, then the degree of polynomial will be zero; otherwise, the degree will be greater than zero and will depend on the pattern of homogeneity. This fact is exploited by constructing the orthogonal polynomial regression of estimated slopes on the adjusted intercepts, and then testing homogeneity of slope by testing the significance of the individual coefficients of the orthogonal polynomial. Such a test procedure amounts to partitioning the ordinary F-test " among regression coefficients " into at most $(t-1)$ individual tests designed to have greater power against specific patterns of heterogeneity. Mention is made that this procedure compares with Tukey's single degree of freedom tests for non-additivity, and the mathematical justification follows that outlined by Tukey [*Biometrics* (1949) **5**, 232-242].

The theory for testing the polynomial regression of slope on adjusted treatment mean is developed, and a numerical example showing the computational procedure is illustrated. Finally, mention is made of the dual result for testing homogeneity of intercepts. Here the assumption of constant slopes and variable intercepts is replaced by the dual assumption of constant intercepts and variable slopes.

It should be noted that under certain conditions, the test developed in this paper also has power against non-linear alternatives to the hypothesis of parallel straight lines. Some caution must therefore be exercised in interpreting a significant F-value in this (or any other) test of linear homogeneity.

(W. H. Beyer)

2/535

WILKINSON, G. N. (Commonw. Sci. Industr. Res. Organisation, Adelaide) **7.4** (7.6)
Comparison of missing value procedures—*In English*
Aust. J. Statist. (1960) **2**, 53-65 (12 references, 5 tables)

When loss or rejection of some of the observations introduces fortuitous lack of orthogonality into the data of a designed experiment, a familiar technique of analysis is to replace the missing values by estimates determined by the method of least-squares. When more than one value is missing, the estimates are often determined by iterative methods. Alternatively, an adaptation of covariance analysis, with dummy variates to represent the missing values, may be used.

In this paper, the writing down and solving of the simultaneous equations for the estimates, called the " direct procedure " is advocated. This method is set out in matrix terms, and then compared with the covariance procedure. The two procedures are shown to be equivalent, and the correspondences between them are set out. In general, the direct procedure requires fewer steps, and in particular gives the correct residual sum-of-squares directly.

The iterative procedure is also discussed and compared with the direct procedure. The advantages of determining explicitly the equations for the missing values are that (i) they enable exact standard errors for comparisons affected by the missing values to be calculated, and (ii) they enable the most efficient and concise method of solution to be found.

(E. J. Williams)

2/536

GALANTINO, F. (School of Statistics, Palermo) **8.1** (8.7)

Statistical method in the survey of social phenomena. Recent contributions to some sampling techniques—*In Italian*

Ann. Fac. Econ. Com. Palermo (1959) **13**, 1-98 (91 references, 2 tables)

In this paper the author analyses some theories of sampling and, principally, the sampling theory for estimates based on fewer individuals than the number selected (Durbin), the theory of stratified and casual samples (Deming, Yates), the estimation of an optimum sub-sampling number (Brooks), and so on.

He proposes some sub-sampling methods; with reference to the Brooks theory, he considers in the general scheme

$$y_{ij} = \bar{Y} + u_i + e_{ij}$$

where y_{ij} is the value of the jth element of the ith unit, $\bar{Y}$ the population mean, u_i the deviation of the mean of the ith unit from the population mean, e_{ij} the deviation of the jth element of the ith unit from the mean of the ith unit, the hypothesis

$$C/C_u = \bar{\sigma}_u^2/\bar{\sigma}_y^2 = K$$

where C is the cost of executing the two-stage sample, C_u the cost of selecting a single unit for sub-sampling, σ_u^2 the variance of u_i, $\bar{\sigma}_y^2$ the variance of $\bar{Y}$, K being a constant.

The number n of units selected for sub-sampling is

$$(1+\bar{\sigma}_e^2/m\bar{\sigma}_u^2)(1+mC_e/C_u),$$

m being the number of elements sampled within each selected unit or sub-sampling number, $\bar{\sigma}_e^2$ the variance of e_{ij} and C_e the cost of sampling an element within a selected unit.

The optimum sampling number (n_1 of n) may be calculated when the optimum sub-sampling number (m_1 of m) and the ratios between variances and between costs are known.

The author, however, generalises the known scheme $C = nC_u + nmC_e$ by proposing the relation $C = nC_u + n\bar{m}C_e$ where

$$\bar{m} = \sum_{v=1}^{s} m_i/S$$

is the mean of sub-sampling numbers.

Generalising the questionnaire scheme of Gallup, the author shows some new practical schedules for social survey, with reference to problems of sampling procedures for mailed questionnaires (El Badry).

2/537 (F. Galantino)

JÁNOSSY, L. & RUPP, E. (Hungarian Academy of Sciences, Budapest) **8.1** (4.3)

Determination of parameters in case of exponential decay—*In Hungarian*

Magy. Tud. Acad. Központi Fiz. Kut. Intézet. Közl. (1960) **8**, 75-81 (3 references, 3 figures)

In this paper the authors divide the measuring time into several intervals and estimate the parameters of the exponential decay by means of the numbers of particles arriving during each interval. The suitable choice of the dividing points is considered. This is shown to be essentially equivalent to the special case of the optimum one-dimensional stratification problem (proportionate allocation) of Dalenius [*Sampling in Sweden* (1957) Uppsala: Almqvist & Wiksell].

A graph is given, representing the suitable stratification points as function of the first stratification point. If the preliminary value of the life-time parameter is known, the stratification points can be determined, by use of this graph, for given measuring time and number of intervals. In a special case, namely for the division of the interval $(0, \infty)$ into two intervals, they obtain the same result as Dalenius.

Investigations concerning the efficiency lead to the conclusion that a measure-time which is ten times the mean life-time and its division into four intervals are, in practice, sufficient.

(K. Sarkadi)

2/538

SARKADI, K. (Math. Inst., Hungarian Academy of Sciences, Budapest) **8.8** (2.4)

A rule of dualism in mathematical statistics—*In English*

Acta Math. Acad. Sci. Hung. (1960) **11**, 83-92 (19 references)

Let us denote by ζ the fraction defective, by κ_n the number of defective items found in a sample of size n taken from an infinite population. The "rule of dualism" states a connection between the confidence limits and the inverse probability limits of the fraction defective. In the case of uniform *a priori* distribution this can be expressed in the form

$$\text{(i)}\ \Pr(\zeta<p \mid \kappa_n=k)=\Pr(\kappa_{n+1}\geqq k+1 \mid \zeta=p),$$
$$k=0, 1, 2, \ldots, n;\ 0<p<1.$$

If the lot is finite and contains N elements, then the corresponding relation is as follows:

$$\text{(ii)}\ \Pr\left\{\zeta\leqq \frac{M}{N}\,\middle|\, \kappa_{n,\,N}=k\right\}$$
$$=\Pr\left\{\kappa_{n+1,\,N+1}\geqq k+1 \,\middle|\, \zeta=\frac{M+1}{N+1}\right\}.$$

The above relations are treated by the author who also deals with several generalisations of the binomial case (i) and the hyper-geometric case (ii).

One generalisation concerns the poly-hypergeometric extension of (ii) for the case when more than two alternatives are considered. As a limiting form the polynomial case is obtained. A further generalisation is the Poisson case when, in a continuous production, the number κ of defects in a lot follows the Poisson law. In this case

$$\Pr(\lambda<l \mid \kappa=k)=\Pr(\kappa\geqq k+1 \mid \lambda=l)$$

holds, where λ is the expected value of the Poisson distribution having *a priori* uniform distribution over the half-line $(0, \infty)$, that is on the interval $(0, L)$; $L\to\infty$. The final generalisation concerns the mixture of Beta distributions as the *a priori* distribution.

The paper concludes with some remarks on the same acceptance method of Oderfeld which concern the determination of the parameters of the *a priori* Beta distribution on basis of the past sample results. Some of the results have been previously published in Hungarian [*Magy. Tud. Akad. Alkalm. Mat. Int. Közl.* (1953) **2**, 275-293]. The proofs are based on probabilistic arguments instead of on direct computations.

(I. Vincze)

2/539

THIONET, P. (Inst. Nat. Statist., Études Econ., Paris) **8.3** (4.6)

The adjustment of sampling results to results of counts—*In French*

Rev. Int. Statist. Inst. (1959) **27**, 8-25 (11 references)

This article reverts to a problem already studied by numerous authors from Deming & Stephan [*Ann. Math. Statist.* (1940) **18**, 427-444] through to El Badry & Stephan [*J. Amer. Statist. Ass.* (1955) **50**, 738-762]: the problem considered is how to improve a sample estimate by the supplementary information, from two independent samples, contained in the marginal distributions of the population when sampled with respect to two different stratification variables.

If the population was known in its entirety, this would be a classical case of "re-weighted" estimation, also called stratification *a posteriori*. In attempting to reduce the problem to this case by first reconstructing the distribution from the marginal ones the author has retained two particular cases from all possible distributions as studied by Fréchét.

The first solution of a certain quadratic programming problem, is the one which minimises the variance of the estimator concerned, irrespective of possible bias; the second is that of Morgenstern-Gumbel, adjusted by maximum likelihood methods. Experimentally a close analogy is noted between the results of the second method and the two routine procedures already known: their common fault would be that they do not produce a distribution which resembles the sample. The same could be said of any other method: and one is led to suppose that this absence of resemblance is due to deformations of the sample. This paper departs from the classical theory of "good" samples and gives procedures for treating "bad" samples: it is essential that hypotheses about the nature of the deformations should not be in contradiction with hypotheses on which any given method is based.

The first method seems to be appropriate if the deformations of the rate of sampling by non-response vary much from one cell of the contingency table to another. On the other hand, the second, which supposes the sample-units to be independent, would be appropriate if the sample is too small without being systematically deformed with respect to the stratification variables. In addition a third method is studied which supposes that all characteristics are in a linear regression with the two stratification variables. The separate samples of these variables allow of an estimation by regression: the regression plane being estimated from the sample.

(P. Thionet)

2/540

BRADLEY, R. A. & PENDERGRASS, R. N. (Virginia Poly. Inst., and Radford Coll., Va.) **9.5** (9.0)
Ranking in triple comparisons—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 229-241 (17 references, 5 tables)

A mathematical model for triple comparisons has been developed as an extension of the earlier developed model for paired comparisons [see Bradley & Terry, " The rank analysis of incomplete block designs. I. The method of paired comparisons ", *Biometrika* (1952) **39**, 324-345]. Parameters $\pi_1, \ldots, \pi_t$ are associated with probabilities of rankings of t treatments in n repetitions of the possible $\binom{t}{3}$ triple comparisons.

Based on the model postulated, tests of treatment equality are outlined and a test of agreement of treatment rankings over sets of repetitions of triple comparisons is given with tests of the appropriateness of the model.

Estimation is based on maximum likelihood procedures and large-sample distributions of estimators, and test statistics are based on likelihood ratio theory. The suitability of these large-sample distributions as approximations for moderate size samples is considered.

A brief example illustrates the techniques developed.

(R. A. Bradley)

2/541

DeBAUN, R. M. & CHEW, V. (Amer. Cyanamid Co., Stamford, Connecticut) **9.4** (7.3)
Optimum allocation in regression experiments with two components of error—*In English*
Biometrics (1960) **16**, 451-463 (2 references, 6 tables, 1 figure)

In the literature, experimental designs for regression analysis are usually optimised with respect to the total number of observations to be taken. In this paper cost functions are introduced and optimum designs are derived for both extrapolation and interpolation, including the case where " split-plotting " occurs.

The authors discuss the situation in which replication in full plots are very expensive but replications within plots are relatively cheap, as they might be when aliquots of experimental material are the within-plot elements and whole lots of experimental material represent the whole plots. The number of sub-plots within the whole plots for greatest efficiency (in the sense of most information per unit cost) depends on the magnitude of between- and within-plot errors as well as on the incremental cost of additional within-plot replications. This general subject has been studied extensively in terms of the usual experimental designs but this paper is primarily concerned with the problem of regression experiments, particularly those in which extrapolation is the principal purpose. Attention is called, by the authors, to the common use of the method of " extrapolation to infinite dilution " used to find the molecular weight of polymeric substances.

The primary section presents the basic development which includes expressions and tables for optimum group size and efficiency for estimation of means in terms of the ratio of the between-plot variance to the total variance and the proportional cost of taking an additional sub-plot observation. It is also noted that the efficiency function is relatively insensitive to variations in plot size in the neighbourhood of the optimum. The authors then go on to develop expressions to specify the efficiency and the most efficient designs for both interpolation and extrapolation for the following cases:

(i) Regression on one independent variable—Independent observations,
(ii) Regression on one independent variable—Non-independent observations,
(iii) Bilinear regression—Non-independent errors.

A numerical example of last case is presented to illustrate various features of the technique.

(R. J. Taylor)

2/542

RAO, C. R. (Indian Statistical Institute, Calcutta) **9.0** (7.6)

Experimental designs with restricted randomisation—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 397-404 (2 references). Discussion abstracted No. **2**/544.

In this paper some designs for experiments involving several groups of treatments, providing precisions of different order for within group and between group comparisons are discussed.

The principle adopted is that of restricted randomisation. The treatments whose differences are to be estimated with higher precision, occupy neighbouring (contiguous) plots. Thus instead of randomising all treatments over all the plots in a block, the block is split up into sub-blocks of contiguous plots, the groups of treatments are assigned to sub-blocks at random and within each sub-block the treatments within a group are randomised over the plots. Here the within-group treatment comparisons are estimated more precisely than the between-group treatment comparisons. Some other methods of restricting randomisation to attain similar results are also suggested.

Under this type of randomisation, the analysis is derived for the following cases:

(i) When two groups of v treatments each are arranged in randomised blocks of size $2v$.

(ii) When two groups of v_1 and v_2 treatments respectively, are arranged in blocks of size $2k$ so that each block contains k treatments from each group and any two treatments from the ith and the jth group occur in λ_{ij} blocks $(i, j) = 1, 2$.

(K. R. Shah)

2/543

RAO, C. R. (Report of discussion on a paper by) **9.0** (7.6)

Experimental designs with restricted randomisation—*In English*

Bull. Int. Statist. Inst. (1960) **37**, I 56-57

Those taking part in this discussion were Finney, Madow and Tippett: the author briefly replied.

Finney asked for further information on the question of the method of comparing two experiments: for example, was there a single scale of money or some nutritional standard? The author's reply was to the effect that the experiments were designed primarily for evaluation of the response surfaces as a function of the proportion of mixture. Once these were available other things could be studied such as the effect suggested by Finney.

Madow suggested that the response surface type of design seemed to be most appropriate and Tippett commented that there would be some difference if the mixed crops were harvested separately or in their mixed form: the author confirmed that the harvesting was separate.

(W. R. Buckland)

2/544

RÜMKE, C. L. (Dept. Pharmacol. and Neurobiol., Free Univ., Amsterdam) **9.5** (5.7)

An efficient design for comparing the effects of two treatments—*In English*

Symp. Quantative Methods in Pharmacology (1960)

The variability of the reactions of groups of animals to drugs often interferes with an efficient comparison of the effects of two treatments with all-or-none responses. An efficient design for an experiment in which the probabilities of an effect after two treatments may be compared has been derived from Dixon and Mood's " up-and-down " method. Two groups of animals which received the two different pre-treatments to be compared are treated with logarithmic equidistant doses of a convulsant agent. In each group the series of experiments is interrupted as soon as the sign of the effect has changed three times. The average of the two means of the last two doses administered is calculated. Thereafter *n* animals of group I and *n* animals of group II are given this average dose. A test for the difference between the probabilities of an effect after the two pre-treatments is performed in a fourfold contingency table at a dosage at which the proportions of animals showing an effect may be expected to be distributed approximately symmetrically round 0·5.

This " semi-sequential " method leads to a test for the comparison of the effects of two treatments with all-or-none responses. However, an estimate of the difference between the ED_{50}'s cannot be obtained by this method. This opportunity arises by the following modification of this design—though with some loss of efficiency.

The experiment is started in the way described. The average of the two means of the last two doses administered is determined. Thereafter two doses are chosen, which have about the same difference from this average. In group I as well as in group II $\frac{1}{2}n$ animals are to be treated with the lower dose and a same number of animals is to be treated with the higher dose. A combined test is performed on the two fourfold contingency tables which are to be constructed from the results. From the proportions of successes found after the different dosages, the ED_{50}'s can be obtained with the aid of a probability paper or by some calculation using the *arc sin* root transformation.

(C. L. Rümke)

AKAIKE, H. (Institute of Statistical Mathematics, Tokyo) **10.6** (–.–)

On a limiting process which asymptotically produces f^{-2} spectral density—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 7-11 (4 references)

In some of the reports about the spectral analyses of roughness of runways or roadways the spectral densities approximately of the form c/f^2 (f: frequency, c: constant) are observed. In this paper a simple model is tentatively suggested to explain how this $1/f^2$ phenomenon takes place.

The model shows that by applying the roughening and smoothing operations alternatively and repeatedly to the stochastic process, which is taken to represent the profile of a runway or roadway, the spectral density function of the process tends to some limiting form. In this model the roughening operation is represented as an addition of a stationary stochastic process of fixed type to the former profile and the smoothing operation as an application of a convolution transformation to the profile. By taking into consideration the physical meanings of roughening and smoothing, some natural assumptions are imposed on the form of the spectral density of the process which is added to the profile and on the form of the smoothing function to get the $1/f^2$ form. This approximation of the $1/f^2$ form is only valid in the range of $f>f_0>0$ (f_0: arbitrary but fixed).

During the discussion of the model the author indicates the necessity of the adoption of some trend elimination procedure in the spectral analyses of runway or roadway profiles.

(H. Akaike)

2/547

BELYAEV, Y. K. (Moscow) **10.1** (–.–)

An example of stochastic process with mixing—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 101-103 (3 references)

The example given in this paper is a stochastic process $\xi(t)$ with a continuous parameter and mixing, whose range consists of two states. It is shown that the integral $\zeta(p) = \int_0^p \xi(t)dt$ does not have any increase in variance.

(Y. K. Belyaev)

2/548

BEN-ISRAEL, A. & NAOR, P. (Technion, Israel Inst. Tech., Haifa) **10.4** (2.5)
A problem of delayed service. I.—*In English*
J. R. Statist. Soc. B (1960) **22**, 245-269 (4 references, 2 tables)

A system is considered in which an operator is in charge of a number of machines which are liable to break down. The operator walks along the machines in a prescribed order, attends to them, and whenever he finds a broken-down machine he restores it to a working condition.

It is assumed that attendance time at a machine is independent of whether the machine is working or not. This means that repair time is negligible compared to time spent either walking from machine to machine or on inspection and maintenance of all machines. Two forms of breakdown distribution are considered: firstly, Poisson I where the time elapsing between successive breakdowns in the system is a random variable possessing an exponential distribution. Secondly, Poisson II where the uninterrupted working of a machine is a random variable with an exponential distribution.

Two types of models are considered:

(i) Simultaneous models where the operator arrives at the set of machines and those machines which are broken-down are repaired within a time very much shorter than the time elapsing between successive arrivals of the operator.

(ii) Successional models where the machines are inspected, and possibly serviced, in some prescribed order.

Distribution functions are defined for the inter-arrival times of the operator, the " round-time distribution " and for the time that elapses from the actual breakdown of a machine up to the time of repair. The relation between these two is found for simultaneous models with both Poisson I and II forms of breakdown. For successional models only the Poisson II case is treated fully, whilst some of the assumptions are slightly modified for Poisson I breakdowns. Three particular forms of round-time distribution are considered, exponential, constant and Erlangian.

Two further random variables are considered; the number of idle machines as observed by a controller who observes the system at random instants, and the number of broken machines encountered by a repairman during one round. The means and variances for these variables are given for most of the situations.

See also abstract No. **2**/550.

(C. Burrows)

2/549

BEN-ISRAEL, A. & NAOR, P. (Technion, Israel Inst. Tech., Haifa) **10.4** (2.5)
A problem of delayed service. II.—*In English*
J. R. Statist. Soc. B (1960) **22**, 270-276 (2 references)

This is a continuation of the previous paper [*J. R. Statist. Soc.* B (1960) **22**, 245-269; abstracted in this present journal No. **2**/549, **10.4**] and presents a model which is intermediate to those considered earlier.

It is assumed that the machines are attended to in groups, repairs of machines in the same group being dealt with in succession. Both Poisson I and II forms of breakdown are considered.

Expressions are developed for the mean and variance of the number of idle machines as observed by a random controller and as observed by the repairman during one round.

(C. Burrows)

2/550

BIRNBAUM, Z. W. (Statist. Res. Lab., Univ. Washington, Seattle) **10.1** (10.4)

Life length and failure of materials interpreted as stochastic processes—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 214-217 (1 reference). Discussion abstracted No. **2**/552

A stochastic model is presented for life-lengths of structures which are subject to failure attributed to fatigue, brittle failure and similar phenomena.

The model in this paper assumes that the failure rate of the structure at any time τ is determined by the state of the structure $S\tau$ and the instantaneous damage $\Delta\tau$ affecting the structure at that time, and that either $S\tau$ and $\Delta\tau$ may be a fixed function of time or a stochastic process. It is shown how, under certain conditions, the probability distribution of the life-length of a structure can be derived from information on $S\tau$ and $\Delta\tau$.

The results of these derivations in specific cases are reported: see also Birnbaum & Saunders [*J. Amer. Statist. Ass.* (1958) **53**, 151-160].

(Z. W. Birnbaum)

2/551

BIRNBAUM, Z. W. (Report of discussion on a paper by) **10.1** (10.4)

Life length and failure of materials interpreted as stochastic processes—*In English*

Bull. Int. Statist. Inst. (1960) **37**, I 99

Those taking part in this discussion were Gumbel and Hemelrijk: the author replied briefly.

Gumbel stated that most physicists believed in a difference in principle between static and dynamic loading. The irreversible molecular changes in the latter gave rise to the question as to whether the existence of the minimum life and endurance limit was foreseen. The author replied that it was possible to obtain these in accordance with the model proposed in the paper but it was difficult to decide whether material counterparts existed.

Hemelrijk asked for clarification of certain symbols in the development of the argument and whether some variables were random or not. The author referred to a paper [*J. Amer. Statist. Ass.* (1958) **53**, 151-160] and clarified the detail of the points raised.

(W. R. Buckland)

2/552

BLOEMENA, A. R. (Mathematical Centre, Amsterdam) **10.4** (10.1)

On queueing processes with a certain type of bulk service—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 219-227 (13 references)

Customers arrive at a counter to be served and line up in the order of their arrival. The time between two successive arrivals has an exponential distribution with unit mean. A batch of (at most) n customers is served at the same time. The service times, which are supposed not to depend on the size of the batch, are identically distributed with mean μ. All inter-arrival times and all service times are mutually independent. Service is stopped when no customers are present and resumed at the moment a new customer arrives, a situation which often occurs in practice; for example, in the case of lifts, shiplocks, etc. In this respect the situation differs from that studied by Bailey [*J. R. Statist. Soc.* B (1954) **16**, 80-87] and Downton [*J. R. Statist. Soc.* B (1955) **17**, 256-261; and (1956) **18**, 265-274], where service is supposed to be continued even when no customers are present.

The queue-lengths at the moments that the serving of a batch is just completed constitute a Markoff chain, which is seen to be irreducible and aperiodic. The chain is ergodic if $\mu < n$.

The stationary state distributions of the queue-length and the waiting-time are studied. The generating function of the queue-length distribution and the Laplace transform of the waiting-time distribution are obtained by interpreting these functions for values of the argument between zero and unity as probabilities.

(F. W. Steutel)

2/553

BRODY, S. M. (Kiev) **10.4** (2.5)

On an integro-differential equation for systems with τ-waiting—*In Ukrainian*

Dopov. Acad. Nauk. Ukrain, SSR (1959) 571-573 (3 references)

The author considers the Poisson's flow of demands on the service of one instrument. The time of service is a random variable having exponential distribution. The demand entering the system is served immediately if the instrument is not engaged and has to wait in line if the instrument is engaged. The demand waits for service no longer than the time τ, after which it is lost. Integro-differential equations for the time of waiting for service are obtain in the paper.

(S. M. Brody)

2/554

CHIANG, C. L. (University of California, Berkeley) **10.9** (2.4)

A stochastic study of the life table and its applications. I. Probability distributions of the biometric functions—*In English*

Biometrics (1960) **16**, 618-635 (12 references)

This is the first of a series of papers which will deal stochastically with life tables by treating the biometric functions in the tables as random variables. In this investigation, distributions, expected values and co-variances of these functions are derived. A second paper of the series will be devoted to the derivation of formulas for the sample covariances. A third paper will be concerned with applications to studies of patients afflicted with certain diseases.

The author discusses briefly the two forms of life tables in common use: (i) the cohort (generation) life table and (ii) the current life table. In the derivation of the distribution of l_x, the number of individuals surviving the age interval $(0, x)$, the age x is treated as a continuous variable. Consequently, based on the derived distribution, a formula is obtained for the probability of l_i individuals surviving any specified age interval in the life table, $(0, x_i)$. The joint distribution of the random variables $l_1, l_2, \ldots, l_w$, given that there were l_0 individuals at age 0, is shown to be the product of w binomial distributions: and the author shows that the number of deaths $d_0 = \delta_0$, $d_1 = \delta_1, \ldots, d_w = \delta_w$ in the life table have a multinomial distribution with probabilities of $p_{0i}q_i$ of an individual dying in the ith interval (x_i, x_{i+1}); p_{0i} being the probability of surviving the age interval $(0, x_i)$, q_i the probability that a member of age x_i will die in the ith interval, and l_0 the size of the original cohort.

The expectation of the proportion of deaths in the age interval (x_i, x_{i+1}) is given in the fifth section. For large values of l_0, an approximate formula is derived for the variance of the observed proportion of survivors, $\hat{p}_i$. The author shows that the covariance between the proportion of survivors in two disjoint age intervals is zero but points out that $\hat{p}_i$ and $\hat{p}_j$ are not independent.

The distribution of $\hat{e}_\alpha$, the observed expectation of life at age x_α is discussed: in addition to the general expression for $\hat{e}_\alpha$, the computational formula for $\hat{e}_\alpha$ is given based on the assumption that the distribution of deaths in each interval is uniform. A theorem states that as l_α increases the distribution of the observed expectation of life at age x_α is asymptotically normal: formulæ for the mean and variance of this asymptotic distribution are given.

(B. J. Trawinsik)

2/555

ČISTYAKOFF, V. P. (Moscow) **10.1** (-.-)

Transient phenomena in branching stochastic processes—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 31-46 (7 references)

Let $\mu_k(t) = \{\mu_{k1}(t), \ldots, \mu_{kn}(t)\}$ be a branching process with n types of particles and let

$$\Pr\{\mu_{kj}(t) = \omega_j;\ j = 1, \ldots, n\} = \delta_k^\omega + p_k^\omega t + o(t) \qquad (k = 1, \ldots, n)$$

when $t \to 0$. Here $\omega = \{\omega_1, \ldots, \omega_n\}$, $\delta_k^\omega = 1$ for $\omega_k = 1$, $\omega_j = 0$ $(j \neq k)$ and $\delta_k^\omega = 0$ in other cases. Having defined the generating functions by

$$f_k(x_1, \ldots, x_n) = \sum_\omega p_k^\omega x_1^{\omega_1} \ldots, x_n^{\omega_n} \quad (k = 1, \ldots, n)$$

the author denotes the factorial moments by

$$a_{kj} = \left.\frac{\partial f_k}{\partial x_j}\right|_{x=1}, \quad b_{ij}^{(k)} = \left.\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right|_{x=1},$$

$$c_{ijl}^{(k)} = \left.\frac{\partial^3 f_k}{\partial x_i \partial x_j \partial x_l}\right|_{x=1}.$$

Let $\mathfrak{A}$ be the compact set of an undecomposable matrix $\boldsymbol{a} = \| a_{kj} \|$ $(k, j = 1, \ldots, n)$; $\lambda = \max\limits_{1 \le i \le n} (\mathrm{Re}\, \lambda_i)$ where the numbers λ_i satisfy the equality $|\, a - \lambda_i \boldsymbol{E} \,| = 0$ ($\boldsymbol{E}$ being the unity matrix) and $v = \{v_i\}_{i=1}^n$, $u = \{u_i\}_{i=1}^n$ satisfy the equalities $\boldsymbol{a}u = \lambda u$, $v\boldsymbol{a} = \lambda v$,

$$\sum_{k=1}^n v_k^2 = 1, \quad \sum_{k=1}^n u_k v_k = 1.$$

Let $K(\mathfrak{A}, \delta, B, c)$ be a class of $\{f_k(x)\}$ with $\boldsymbol{a} \in \mathfrak{A}$,

$$0 < \delta < \sum_{i,j,k=1} b_{ij}^{(k)} < B < \infty, \quad c_{i,j,l}^{(k)} < c < \infty.$$

The following asymptotic formula for $t \to \infty$, $\lambda \to 0$ holds true uniformly for all $\{f_k\} \in K$

$$1 - \Pr\{\mu_{ij}(t) = 0, j = 1, \ldots, n \mid \mu_i > 0\} \sim u_i k(t, \lambda, 0),$$

$$\mu_i = \sum_{j=1}^n \mu_{ij}(t).$$

The probability distributions

$$S_k^{(t)}(y_1, \ldots, y_n) = \mathrm{r}\left\{\frac{\mu_{kj}(t)}{M\{\mu_{kj} \mid \mu_k > 0\}} < y_j\ j = 1, \ldots, n \,\middle|\, \mu_k > 0\right\}$$

converges to an exponential distribution as $t \to \infty$, $\lambda \to 0$ uniformly for all $\{f_k\} \in K$.

(V. P. Čistyakoff)

CONOLLY, B. W. (Saclant ASW Res. Center, La Spezia, Italy) **10.4** (2.5)

Queueing at a single serving point with group arrival—*In English*

J. R. Statist. Soc. B (1960) **22**, 285-298 (8 references, 4 tables)

This paper studies the effect on a single server queueing system of arrivals taking place in groups of a fixed size instead of singly. The time intervals separating the arrivals of successive groups are independent samples from a fixed distribution. The service time of each individual is assumed to be negative exponentially distributed. The groups are dealt with on a first come, first served basis, though the order of service of individuals within a group is indifferent. A steady state is also assumed. This system is formally equivalent to the queue given in Kendall's notation as $GI/E_c/1$.

By considering the instants of arrivals of groups, a set of integro-difference equations for the state probabilities is set up. These are solved using Laplace transforms and the class of the inter-arrival distribution is restricted to those which give a particular form to this solution. Formulae are then derived giving the distribution of queue length, the mean idle time of a server, the mean waiting time for a customer and also a generating function for the joint probability and probability density $\rho_n(t)$ that a busy period, that is when the server is fully occupied, lasts for time t and then n groups are served.

A numerical example is given with group sizes of 1, 2 and 4 but with the proviso that the arrival intensity for individuals remains constant. The inter-arrival intervals are taken as constant. This shows that when the group size varies but the mean number of arrivals in unit time and the average length of a service period remain constant then:

(i) average queue size increases with increasing group size;

(ii) the average non-zero idle period of the server increases with increasing group size, but if the average is calculated over all occasions when a customer completes service it remains constant;

(iii) the average waiting time of the first member to be served of a newly arrived group increases when the average only takes into account non-zero waiting times, but decreases when it includes all occasions when a group arrives;

(iv) the average number of groups served during a busy period decreases with increasing group size, though the number of customers served increases.

2/557 (C. Burrows)

CSISZAR, I. (Eötvös University, Budapest) **10.5** (1.5)

Some remarks on the dimension and entropy of random variables—*In English*

Second Hungarian Mathematical Congress (1961) (2 references)

The dimension of a random variable ξ is defined as the limit $d(\xi) = \lim_{n\to\infty} H_0(\xi_n)/\log n$, provided this exists; where $H_0(\xi_n)$ denotes the Shannon entropy of the discrete random variable $\xi_n = 1/n[n\xi]$. The d-dimensional entropy of ξ is defined by

$$H_d(\xi) = \lim_{n\to\infty} \{H_0(\xi_n) - d \log n\}$$

always provided that the limit exists and it is finite; $[x]$ denotes the integral part of x.

These notions were introduced by Balatoni & Rényi ["Remarks on entropy", *Publ. Math. Inst. Hung. Acad. Sci.* (1956) **1**, 9-40; and also Rényi, "On the dimension and entropy of probability distributions", *Acta Math. Acad. Sci. Hung.* (1959) **10**, 193-215; abstracted in this journal No. **1**/313, **10.5**].

According to theorem 1, if $H_0(\xi_1) = H_0([\xi])$ is finite, there exists the limit

$$\lim_{n\to\infty} \{H_0(\xi_n) - \log n\}$$

which is either finite or equal to $-\infty$. In the proof the authors use only elementary properties of the entropy. If the dimension of ξ is absolutely continuous with the density function $f(x)$ then the above limit is equal to

$$\xi = \int_{-\infty}^{+\infty} f(x) \log \frac{1}{f(x)}\, dx$$

where the integral surely exists: for the case of finite ζ, the equality of these two limit expressions was proved by Rényi.

Theorem two states that if the distribution of ξ is not absolutely continuous then the first limit is equal to $-\infty$, that is to say the one-dimensional entropy of ξ does not exist. For the proof it is shown that if the distribution of ξ is not absolutely continuous then there is an $\varepsilon>0$ such that for every $\delta>0$ and for every $n>n_0(\delta, \varepsilon)$ there exist such integers k_j, $j=1, \ldots, s$, that $s<n\delta$ and $\Sigma \Pr(\xi_n = k/n)>\varepsilon$.

The sum representing $H_0(\xi_n)$ is composed of two parts, the first of which consists of the members corresponding to the k_j-j; $j=1, \ldots, s$. Using the Jensen inequality we obtain an estimation which proves the statement.

The second theorem cannot be sharpened:

$$H_0(\xi_n) - \log n$$

may tend arbitrarily slowly to $-\infty$ for appropriate singularly distributed random variables. The results obtained can also be generalised for random vectors.

2/558 (I. Csiszar)

DOROGOVTSEV, A. I. (Kiev) **10.2** (1.6)

Statistical analysis of a stochastic difference equation—*In Ukrainian*

Dopov. Acad. Nauk. Ukrain, SSR (1959) 120-124 (1 reference)

The author considers a random process x_t, where t is an integer satisfying the difference equation

$$x_t+a_1x_{t-1}+\ldots+a_px_{t-p}+a_0 = b_0\xi_t+b_1\xi_{t-1}+\ldots b_v\xi_{t-v}$$

where a_i and b_i are unknown parameters and ξ_t are independent variables with a mean equal to zero and a variance equal to unity.

The parameters a_i, b_i are estimated by a sequence of observations $x_1, x_2, \ldots, x_N$ of the process. The case of biased estimates is discussed and estimates of the a_i-parameters are given, these estimates being consistent and their joint distribution tending toward the normal law when $N\to\infty$.

(A. I. Dorogovtsev)

2/559

DREMICK, R. F. (Bell Telephone Labs., Murray Hill, N.J.) **10.1** (1.2)

Mathematical aspects of the reliability problem—*In English*

J. Soc. Indust. Appl. Math. (1960) **8**, 125-149 (18 references)

This paper deals with certain theoretical problems encountered in the study of equipment reliability. The primary purpose of the paper is to report or attempt to establish some uniformity of approach to the reliability problem. A new concept is introduced which is more general than those in common use but which specialises to the usual uses under appropriate conditions.

In the author's formulation it is visualised that an enterprise is to be undertaken which consists of operating a piece of equipment or a group of equipments of the same kind, for a certain period of time. The entrepreneur expects a certain economic return from it, but he also expects his return to be adversely affected by the incidence of malfunction. He will have established in his mind an exact idea of the undesirability to him of every kind of poor performance of his equipment. He can then define the reliability of his equipment in this operation as the mean gain, or loss, which he can expect of it. In this paper this mean gain is adopted as the definition of reliability. The paper brings together many of the concepts and problems of reliability and devotes considerable space to the relationship with renewal theory.

(W. T. Wells)

2/560

FINCH, P. D. (London School of Economics) **10.4** (2.5)

On the transient behaviour of a simple queue—*In English*

J. R. Statist. Soc. B (1960) **22**, 277-284 (5 references)

A single server queueing process is considered with Poisson input and a general distribution of service time. The assumption is made initially about the queue discipline although it is assumed that the server is never idle when there are customers to be served.

The generating function for the probability distribution of queue size after the nth departure from the system is found. An explicit solution is given for the case where the service-time distribution is a negative exponential. The relation between queue size on an arrival and a departure is also considered for first come, first served queue discipline. Limiting distributions are given for both negative exponential and general service-time distributions.

(C. Burrows)

2/561

GHOSAL, A. (Central Fuel Research Institute, Bihar, India) **10.4** (2.5)

Problem of emptiness in Holdaway's finite dam—*In English*

Bull. Calcutta Statist. Ass. (1960) **9**, 111-116 (6 references)

A probability theory of dams has been developed by Moran, Prabhu, D. G. Kendall, Gani and others: see Moran [*Theory of Storage* (1959) London: Methuen]. In this paper, the probability $V(u)$ of the dam drying before it fills up, for a given initial level u, has been derived for Holdaway's finite dam, assuming the input to be exponential.

If x_t be the input during the interval $(t, t+1)$, and z_t the content of the dam at time t, the release scheme of Holdaway's dam is as follows: given $l<k$, $d<m$

(i) If $z_t+x_t \geqq k+m$, release z_t+x_t-k

(ii) If $l+m \leqq z_t+x_t \leqq k+m$, release m

(iii) If $l+d \leqq z_t+x_t \leqq +m$, release z_t+x_t-l

(iv) If $d \leqq z_t+x_t \leqq l+d$, release d

(v) If $z_t+x_t \leqq d$, release x_t+z_t so that $z_{t+1} = 0$.

It is shown that $V(u)$ satisfies an integral equation and a method of solving the equation is discussed. The results, however, are not in a form which can be immediately applied in practice.

(C. S. Ramakrishnan)

2/562

FURSTENBERG, H. & KESTEN, H. (Princeton University) **10.1** (1.5)

Products of random matrices—*In English*

Ann. Math. Statist. (1960) **31**, 457-469 (4 references)

The authors consider a stochastic process $X^1, X^2, \ldots$ with values in the set of square matrices with real or complex entries of a fixed dimension and they investigate the asymptotic behaviour of the product ${}^nY^1 = X^nX^{n-1}\ldots X^1$. They generalise a result and prove a conjecture of Bellman [*Duke Math. J.* (1954) **21**, 491-500] that these products obey a central limit theorem. By taking the norm $\| X \| = \max_i \sum_j | X_{ij} |$, where the subscripts refer to the entry of the matrix, it is proved for a stationary process that the limit as $n \to \infty$ of $\alpha_n = 1/n\{\mathscr{E} \log \| {}^nY^1 \|\}$ exists; call it E. If, additionally the process is metrically transitive and

$$\mathscr{E} \max \{\log \| X^1 \|, 0\} < \infty$$

then also $\beta_n = 1/n \| \log {}^nY^1 \| \to$E. The limiting result for α_n is proved straightforwardly. The result for β_n is obtained in two stages. First, showing that the lim sup $\beta_n \leqq$ E and the second is accomplished by considering an auxiliary process $(\boldsymbol{X}_1, \boldsymbol{Z}_1), (\boldsymbol{X}_2, \boldsymbol{Z}_2), \ldots$, where $\boldsymbol{X}_n$ are matrices in the range of the X^n and $\boldsymbol{Z}_n$, are matrices satisfying a recursion condition involving the norm and such that the projection of the first coordinate is a member of the sample space of the X-process. The idea is to obtain uniform integrability of max $\{\log \| \boldsymbol{X}^2\boldsymbol{Z}^1 \|, 0\}$ with respect to a mixture of measures induced by translation on the (X, Z)-process. Thus a suitable subsequence $\{v_i\}$ of these mixtures converges weakly to a stationary measure μ for the (X, Z)-process and its projection agrees with the given probability measure on the X-process. The main step of the proof being to prove the relation

$$E = \liminf_i \mathscr{E}(v_i) \| \boldsymbol{X}^2\boldsymbol{Y}^1 \| \leqq \mathscr{E}(\mu) \log \| \boldsymbol{X}^2\boldsymbol{Y}^1 \|$$

where $\mathscr{E}(.)$ represents expection with respect to the measure in parenthesis. This yields $\liminf \beta_n \geqq E$.

Next the authors prove a result on the asymptotic behaviour of the entries. Assume that the possible matrix values M for X^1 satisfy the two conditions $M_{ij} > 0$ and $1 \leqq \max M_{ij}/\min M_{ij} \leqq C < \infty$, where the extrema are both over all i, j, and that if Ω is an event for the X-process not depending upon the first $(m+n)$ coordinates, then

$$| \Pr\{\Omega \mid X^1, \ldots, X^n\} - \Pr\{\Omega\} | \leqq D\lambda^n \Pr\{\Omega\};$$

where D and λ are fixed positive constants and $\lambda < 1$.

2/563

continued

FURSTENBERG, H. & KESTEN, H. (Princeton University) **10.1** (1.5)

Products of random matrices—*In English*

Ann. Math. Statist. (1960) **31**, 457-469 (4 references)

continued

The theorem is: If $\mathscr{E} | \log X_{11}^1 |^{2+\delta} < \infty$ for some $\delta > 0$ then for a, b (their defining equations are given) we have, if $b > 0$, that $\{\log ({}^nY^1)_{ij} - na\}/\sqrt{(nb)}$ converges in law to $\Pi(0, 1)$ and if $b = 0$ then $\{\log({}^nY^1)_{ij} - na\}/\sqrt{n}$ converges in probability to 0. The proof largely follows the treatment in Doob's book, [*Stochastic Processes* (1953) New York: Wiley]. The paper concludes with two examples indicating directions in which generalisations of these results are not possible.

(S. C. Saunders)

GNEDENKO, B. V. (Moscow University) **10.4** (–.–)

On some aspects of the theory of queues—*In German*

Math. Tech. Wirtschaft (1960) **4**, 162-166 (14 references)

This article presents a review of some results of the theory of queueing, with special regard to some more recent models which have been dealt with by Russian mathematicians.

After a general introduction in which the author emphasises the inter-dependence between theory and practice in the field of mathematics, with quotations by Aristoteles, Tchebycheff, Gauss, etc. he deals with the theory of queueing. The models as considered by Gnedenko refer, above all, to the following generalisation of the simple problem: the units entering the system are not willing to accept unlimited waiting time, but may possibly leave the system without waiting for completion of the service or without even entering the service station. Some more simple models as developed by Barrer, Takács, Jaroschenko and Farinitsch are presented, in which various aspects of deserting are considered.

The author deals with a comprehensive model by Kowalenko, which contains all the cases already considered as special cases. The essential novelty suggested by Kowalenko results from the fact that the time which the individual units are prepared to spend on the service proper is a random variable, the distribution of which depends on the actual waiting time beginning with the time of joining the queue and ending with the entrance to the service point. Further, the time which the units are willing to wait in the queue is a time-independent random variable.

Assuming exponentially distributed service-time and Poisson-input into the queue, an integro-differential equation for $F(x, t)$ is established. $F(x, t)$ indicates the probability that all units arriving before the moment t have left the system in the interval $(t, x+t)$. It can be shown that in the model of Kowalenko $F(x) = \lim_{t\to\infty} F(x, t)$ exists. With the aid of the function $F(x)$ the most important characteristics with respect to the utilisation of the service-procedure can be expressed; for example the distribution of waiting time, of the total time spent in the system, the probability of loss (that is the case of the unit departing without completion of service).

Kowalenko's solution is illustrated by an example applying the simple model of Barrer in which the waiting time spent in the queue has a fixed limit. All the models presented are restricted to service systems with a single channel.

2/565 (F. Ferschl)

GNEDENKO, B. V. (Moscow University) **10.4** (2.5)

On a generalisation of Erlang's formulae—*In Ukrainian*

Dopov. Acad. Nauk. Ukrain, SSR (1959) 347-350 (4 references)

The paper gives a generalisation of Erlang's formulae for the case when devices serving demands may become out of order and require expenditure of the operator's time for their repair or replenishment. The case is considered when the demands are served by n devices, and these devices by r operators. The inflow of demands, as well as the flow of impairments of the devices are assumed to be simple Poisson flows. The probability distribution of the time of service of both the demands and the instruments is assumed to be exponential.

(B. V. Gnedenko)

2/566

GRIFFING, G. (Commonw. Sci. Industr. Res. Organisation, Canberra) **10.9** (-.-)
Accommodation of linkage in mass selection theory—*In English*
Aust. J. Biol. Sci. (1960) **13**, 501-526 (10 references, 1 table)

The purpose of this paper is to develop mass selection theory which will accommodate not only linkage but will provide for different recombination frequencies in the two sexes.

The theoretical aspects of the linkage problem are developed in three stages:

(i) The mass selection theory for two loci is extended to accommodate different recombination values for the two sexes.
(ii) A method is developed by which the generalised two-locus model may be used to cope with genetic situations which are considerably more complex. This method requires the estimation of the recombination value averaged over all possible pairs of loci.
(iii) The expectations of the half-sib and full-sib covariances for a random-mating population are generalised to permit different recombination values for the two sexes. This allows unbiased estimates of genotypic variance components to be obtained.

Finally, application of the more general mass selection theory to the problem of detecting the influence of natural selection in modifying the effectiveness of artificial selection, is discussed.

(G. Griffing)

2/567

HIDA, T. (University of Kyoto, Japan) **10.1** (1.0)
Canonical representations of Gaussian processes and their applications—*In English*
Mem. Coll. Sci., Kyoto (1960) **33**, 109-155 (22 references)

This paper gives systematic establishment of the Lévy's theory, which is concerned with the canonical representation of Gaussian processes, and a development of some new facts. In section one, the general theory of representation is introduced. After showing, as Lévy proved, that a canonical representation of any Gaussian process is uniquely determined if it exists, the author gives a necessary and sufficient condition for the process to have a canonical representation; using Hellinger-Hahn's Theorem in the theory of Hilbert Space.

The author gives a criterion to determine whether a given representation is canonical or not, which is a generalisation of Karhunen's theorem on a stationary process. This criterion is different from Lévy's, which is given with the help of Hellinger integral.

In section two, multiple Markoff Gaussian processes are discussed by using the general theory of representation. In this paper the author generalises the notion of N-ple Markoff processes, which were defined by Doob or Lévy, in order to treat the processes which are not always differentiable. It is shown that the canonical kernel of the N-ple Markoff process is a Goursat kernel of order N, which generalises the fact obtained by Lévy, and special N-ple Markoff Gaussian processes which are differentiable up to $N-1$ times are discussed. As the generalisation of Doob's Theorem, it is proved that the stationary N-ple Markoff process is the sum of general simple Markoff processes. Further, exact forms of predictor are given for certain classes of multiple Markoff processes.

Section three is concerned with Lévy's $M(t)$ process. Let $M_N(t)$ be the spatial average of an ordinary Brownian motion with N-dimensional parameter over the sphere with centre O (origin). Lévy obtained the canonical representation of this process only when N is odd. Simplifying his proof by transforming $M_N(t)$ into a stationary process, the author gives the canonical representation of $M_N(t)$, which can be applicable even when N is even. It is also shown that the number of different representations of $M_{2P+1}(t)$ is just 2^{P-1} including canonical one. Further, it is proved that $M_{2p}(t)$ is not a multiple Markoff process, though $M_{2P+1}(t)$ is a $(p+1)$-ple Markoff process.

(M. Huzii)

2/568

KEMENY, J. G. & SNELL, L. (Dartmouth College, Hanover, U.S.A.) **10.1** (0.6)

Finite continuous time Markoff chains—*In English*

Teor. Veroyat. Primen. (1961) **6**, 110-115

In a recent book [*Finite Markoff Chains* (1959) Princeton: Van Nostrand] the authors developed a method for computing the basic parameters of a finite Markoff chain. Matrix expressions were obtained for these quantities in terms of certain basic matrices easily obtainable from the transition matrix. In this note corresponding expressions are given for the basic quantities for a finite continuous time chain.

(B. V. Gnedenko)

2/569

KHASEN, E. M. (Moscow) **10.1** (10.2)

Evaluation of the single-dimensional probability density of random process in the output of a nonlinear system—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 130-138 (10 references)

In this paper the author considers nonlinear systems with feed-back. These systems contain nonlinear functional reformers, which are separated by linear filters with rational spectra.

In order to evaluate the single-dimensional probability density and the moments of a random process in the output of the system, the author constructs a multi-dimensional random Markoff process; and employs the Kolmogoroff diffusional equations.

The exact results obtained by using this method are compared with the results of an approximate " statistical linearisation " approach for one system. The two sets of results are found to agree well.

(E. M. Khasen)

2/570

KRZYZAŃSKI, M. (Math. Inst., University of Kraków) **10.1** (10.8)

The diffusion under the action of an exterior force and the Kolmogoroff equation—*In French*

Second Hungarian Mathematical Congress (1961) (2 references)

A sequence of stochastic processes relative to a diffusion is considered where an exterior force expressed by a function $f(x)$ of the abscissa X is in action. Systems of registration, moments of the abscissa of a moving particle

$$t_0^{(m)},\ t_1^{(m)},\ \ldots,\ t_n^{(m)} = t$$

correspond to these processes. Let $W_n(x, y, \Delta_k^{(n)})$, where $\Delta_k^{(n)} = t_{k+1}^{(n)} - t_k^{(n)}$, be the probability density of the passage. The function $W_n(x, y, \Delta_k^{(n)})$ is assumed to be given by the formula of Smoluchowski [*Annalen der Physik* (1915) **48**, 1103-1112]

$$\text{(i)}\quad W_n(x, y, \Delta_k^{(n)}) = [4\pi D\Delta_k^{(n)}]^{-\frac{1}{2}} \exp\left\{-\frac{[y-x-\beta\Delta_k^{(n)}f(x)]^2}{4D\Delta_k^{(n)}}\right\};$$

Under certain hypotheses it is proved that the limiting density for $n\to\infty$, $\max\Delta_k^{(n)}\to 0$ satisfies the equations of Kolmogoroff; where the second one is identical to the differential equation of the diffusion under the action of an exterior force. This equation was obtained by Smoluchowski independently from the above formula. A theorem of Khintchine, given in his work [*Asymptotische Gesetze der Wahrscheinlichkeitsrechnung* (1933) Berlin], is applied in the proof. This required the verification of some theorems concerning the properties of the solutions of the parabolic equations.

(M. Kržyzański)

2/571

LEONOFF, V. P. (Moscow) **10.1** (–.–)

On the dispersion of time-dependent means of a stationary stochastic process—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 93-101 (9 references)

Let $\xi(t)$ be a stationary process in the wide sense with discrete (continuous) time $M\xi(t) = 0$

$$\zeta_p = \sum_{t=0}^{p-1} \xi(t) \left(\zeta_p = \int_0^p \xi(t)dt\right),$$

$$b_p = M\,|\,\zeta_p\,|^2.$$

In this paper the author studies the behaviour of b_p for $p\to\infty$.

(V. P. Leonoff)

2/572

MORAN, P. A. P. (Australian National University, Canberra) **10.9** (–.–)

The survival of a mutant gene under selection. II.—*In English*

J. Aust. Math. Soc. (1960) **1**, 485-491 (2 references)

Suppose M haploid individuals are of genotype a or A, with a having a small selective advantage over A so that the expectations of offspring are in the proportions $1+s:1$ respectively. The paper shows that if the population consists initially of k_0 individuals a, the absorption probability P_M that the total population eventually becomes of type a is approximately

$$P_M = (1-e^{-2\theta k_0})/(1-e^{-2\theta M}),$$

where $s/(1+s) \leqq \theta \leqq s$. For $k_0 = 1$, this reduces (when $s^2 M$ is small) to Fisher's result $2s/(1-e^{-2sM})$, [see *The Genetical Theory of Natural Selection* (1930), Oxford: Clarendon Press].

The argument used for this approximation is extended to the case of diploid individuals with distinct sexes, and a similar absorption probability is approximated.

(J. Gani)

2/573

MOTOO, M. (Institute of Statistical Mathematics, Tokyo) **10.0** (1.0)

Diffusion process corresponding to $\frac{1}{2}\Sigma \dfrac{\partial^2}{\partial x^{i2}} + \Sigma\, b^i(x) \dfrac{\partial}{\partial x^i}$—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 37-61 (5 references)

The author first constructs the diffusion process corresponding to the operator

$$A = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{x^{i2}} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x^i}$$

by using the stochastic integral. The method of construction is the multi-dimensional generalisation of that which is treated in Maruyama's paper. The author shows that the semi-group associated with the process can be restricted to certain sub-spaces of continuous functions and it is strongly continuous on these spaces.

Assuming the Lipschitz condition for $b^i(x)$'s, the Hille-Yoshida's generator of the semi-group of the process is investigated. Dynkin's generator of the process is given in the form independent of the process. This gives the exact meaning of the correspondence discussed in constructing the process, and will serve to solve the problems of the differential operator A by the probabilistic method.

Finally, using the fact that the process constructed above and the Brownian motion are absolutely continuous with respect to each other, the author gives some probabilistic properties which are easily derived from those of the Brownian motion.

(M. Motoo)

2/574

MUTH, J. F. (Carnegie Institute of Technology, Pittsburg) **10.2** (6.1)

Optimal properties of exponentially weighted forecasts—*In English*

J. Amer. Statist. Ass. (1960) **55**, 299-306 (12 references)

For time series of the form

$$y_t = \varepsilon_t + \beta \sum_{i=1}^{\infty} \varepsilon_{t-i}, \quad 0 \leqq \beta \leqq 1$$

or of the form

$$y_t = \bar{y}_t + \eta_t$$

where

$$\bar{y}_t = \sum_{i=1}^{t} \varepsilon_i$$

the ε_t are independently distributed with mean zero and variance σ_ε^2 and the η_t are independently distributed with mean zero and variance σ_η^2 it is demonstrated that predictions for the period $t+T$ of the general form

$$y_{t+T}^e = \beta \sum_{i=1}^{\infty} (1-\beta)^{i-1} y_{t-i}$$

are appropriate.

Assuming that α is given by

$$c_t = \alpha \bar{y}_t + \delta_t$$

δ_t distributed independently of $\bar{y}_t$ and y_t^e, an estimate of α may be obtained by replacing $\bar{y}_t$ by the prediction for period $t+1$, y_{t+1}^e, and by applying the ordinary least squares formula, that is

$$\hat{\alpha} = \frac{\Sigma c_t y_{t+1}^e}{\Sigma (y_{t+1}^e)^2}$$

The bias in this estimate is investigated and shown to be small in large samples.

(W. A. Fuller)

ONICESCU, O. (University of Bucharest) **10.5** (6.2)

The entropy-function, correlation and quantity of information—*In French*

Second Hungarian Mathematical Congress (1961)

Let $(\Omega, \mathscr{A}, \mathrm{P})$ be a probability space; Ω is an arbitrary space, $\mathscr{A}$ a σ-algebra of subsets of $\Omega (\Omega \in \mathscr{A})$ and $\mathrm{P}(A)$ a probability measure defined for $A \in \mathscr{A}$. Let Θ be a partition of Ω into a finite number of disjoint sets $\theta_1, \theta_2, \ldots, \theta_n$ $(\theta_k \in \mathscr{A}\ \Pr(\theta_k) > 0)$. The entropy function $H_\theta(A)$ of the partition Θ is defined as follows: $H_\theta(A)$ is a σ-additive set function defined for $A \in \mathscr{A}$ by

$$H_\theta(A) = \sum_{k=1}^{n} \Pr(\theta_k A) \log \frac{1}{\Pr(\theta_k)}.$$

Here the product of two sets denotes their intersection. Clearly $H_\theta(\Omega)$ is equal to the usual entropy of the partition Θ. It is shown that the most important properties of the entropy remain valid for the entropy function too, for example, if $\theta' = (\theta_1', \theta_2', \ldots, \theta_m')$ and $\theta'' = (\theta_1'', \theta_2'', \ldots, \theta_n'')$ are two independent partitions and $\Theta = \theta' . \theta''$ denotes the product of the two partitions, that is the partition formed by the sets $\theta_j' . \theta_k''$ then we have identically for $A \in \mathscr{A}$

$$H_\theta(A) = H_{\theta'}(A) + H_{\theta''}(A).$$

The quantity of information is generalised similarly by defining the information-function $I_{\theta,\phi}(A)$ of the partition $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ with respect to the partition $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$ which is also a set function, and is expressed by the formula

$$I_{\theta,\phi}(A) = \sum_{j=1}^{M} \sum_{k=1}^{m} \Pr(\theta_j \phi_k A) \log \frac{\Pr(\theta_j \phi_k)}{\Pr(\theta_j)\Pr(\phi_k)}$$

Of course $I_{\theta,\phi}(\Omega)$ is equal to the usual quantity of information. The information function for the general, not discrete, case is defined by passing to the limit.

(A. Rényi)

Editorial Note: The original paper used abstract Boolean algebra for the purpose of defining the entropy and information functions.

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.5** (4.0

Some fundamental problems of information theory—*In Hungarian*

Magy. Tud. Akad. III Oszt. Közl. (1960) **10**, 251-282 (24 references)

In this paper the different possibilities of measuring the amount of information are considered. The author investigates whether it is possible to find quantities other than Shannon's measure of information. It is shown that the quantities satisfying certain reasonable postulates concerning the amount of information are only the so-called *measures of information of order* α contained in the value of the random variable having the discrete distribution $P = (p_1, p_2, \ldots, p_n)$, which are defined by $I_\alpha(P) = 1/(1-\alpha) \log_2 \Sigma p_k^\alpha$. The limiting case for $\alpha \to 1$ gives Shannon's measure of information, which is characterised within the family of quantities of information $I_\alpha(P)$ by certain of its properties.

The author investigates the possibilities of extending some known theorems concerning Shannon's measure of information to the measure of information of order $\alpha \neq 1$. He deals with the measures of information concerning the so-called " generalised incomplete-probability distributions " introduced by him; that is, distributions of random variables which are not certainly observable as well as with the relative information of order α. The different definitions presenting themselves concerning the measure of information of order α contained in the observation of a random variable ξ relative to a random variable η are investigated; and taking into account the advantages and disadvantages of these different definitions, it is shown which of them is the most suitable measure of the relative information of order α.

The quantities of information of order α, discussed in this paper have, in several respects, similar properties to the Shannon's measures and they have certain advantages too; for example they are often easier dealt with. The application of the notion of information in statistics is also discussed. It is shown that when dealing with statistical applications the measures of information of order $\alpha \neq 1$ lead to the same conclusions as Shannon's measure of information; for example they are in the same relation with Fisher's " quantity of information ".

(K. Bognár)

2/577

SADAO, I. (College of Science & Engineering, Nihon Univ., Japan) **10.5** (–.–)

A remark on the convergence of Kullback-Leibler's mean information—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **12**, 81-88 (2 references)

Under fairly general assumptions, the author gives two sufficient conditions for the convergence of the mean-information for finite measure. These are generalisations of the conditions given in Lemma 2.1 and Theorem 2.2 of Chapter 4 in Kullback's book [*Information Theory and Statistics* (1959) New York: Wiley].

In this paper the author considers the conditions

$$\text{ess.sup} \left| 1 - \frac{f_i}{f} \right| \to 0, \text{ instead of the condition}$$

$$\lim \left(\frac{f_i}{f} \right) = 1[\lambda]$$

uniformly which was given by Kullback. The author says that, in general, the conditions in Lemma 2.1 and the finiteness of $I(f:g)$ in Theorem 2.2 of Kullback's book are not necessary.

(M. Huzii)

2/578

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.5** (4.0)

On different measures of information—*In English*

Second Mathematical Congress (1961)

Let $[\Omega, \mathscr{A}, P]$ be a probability space. A generalised random variable is defined as an $\mathscr{A}$-measurable function on a sub-set Ω_1 of Ω ($\Omega_1 \in \mathscr{A}$, $\Pr(\Omega_1)>0$). This can be interpreted as the result of an experiment depending on chance which is not always observable, only for $\omega \in \Omega_1$.

Let ξ be a generalised random variable taking on only a finite number of values x_k, $k = 1, 2, \ldots, n$ and B an arbitrary event ($B \in \mathscr{A}$) with positive probability. Put $p_k = \Pr(\xi_k = x_k)$, $q_k = \Pr(\xi_k = x_k \mid B)$ $(p_k>0)$, $\mathscr{P} = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$.

The amount of information concerning ξ, gained by observing the event B is denoted by $I(Q \parallel \mathscr{P})$; it is supposed that this information depends only on $\mathscr{P}$ and Q. Six postulates are given, such that any reasonable measure of information-gain should fulfil all of them. It is shown that if $I(Q \parallel \mathscr{P})$ satisfies these postulates, we have either

$$I(Q \parallel \mathscr{P}) = 1/(\alpha-1)\log_2 \left\{\sum_1^n \left(q_\kappa^\alpha/p_\kappa^{\alpha-1}/\sum_1^n q_\kappa\right)\right\} = I_\alpha(Q \parallel \mathscr{P}) \quad (\alpha \neq 1)$$

$$I(Q \parallel \mathscr{P}) = \left\{\sum_1^n q_\kappa \log(q_\kappa/p_\kappa)\right\} \Big/ \sum_1^n q_\kappa = I_1(Q \parallel \mathscr{P}) = \lim_{\alpha\to 1} I_\alpha(Q \parallel \mathscr{P})$$

$I_\alpha(Q \parallel \mathscr{P})$ is called the "information-gain of order α" concerning the generalised random variable ξ from the observation of the event B; or, from replacing $\mathscr{P}$ by Q. If ξ is an ordinary random variable

$$\left(\sum_1^n p_\kappa = 1, \quad \sum_1^n q_\kappa = 1\right)$$

then $I_1(Q \parallel \mathscr{P})$ reduces to the well-known formula of information gain used in the theory of information.

It is shown that in many respects $I_\alpha(Q \parallel \mathscr{P})$ behaves similarly for all positive values of α. Fisher's measure of information can be deduced from $I_\alpha(Q \parallel \mathscr{P})$ in the same way as from $I_1(Q \parallel \mathscr{P})$. The same remark applies to the distance measure of Mahalanobis, the χ^2-distance, etc.

continued

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.5** (4.0)

On different measures of information—*In English*

Second Mathematical Congress (1961)

continued

An additional postulate is given, which characterises

$I_1(Q \parallel \mathscr{P})$ among all $I_\alpha(Q \parallel \mathscr{P})$.

The "entropy of order α" of the generalised distribution $\mathscr{P} = (p_1, \ldots, p_n)$ is defined in accordance with the above formulae. If ξ is an ordinary random variable, then the expression reduces to the well-known formula, of Shannon for the entropy of a probability distribution. The case of continuous random variables is also discussed together with the connection of entropy of order α with kinetic theory.

(I. Csiszár)

SEREGIN, L. V. (Moscow) **10.1** (-.-)

Continuity conditions of stochastic processes—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 3-30 (8 references)

Let x_t, $0 \leqq t \leqq c < \infty$ be a separable stochastic process in metric space X. The author states that the main purpose of this paper is to derive conditions under which almost all sample functions of process x_t are continuous. We designate $\rho(x, y)$ the distance between the points $x, J \in X$.

Let $P(\ldots)$ be a Markoff transition function, satisfying for each $\varepsilon > 0$

$$\sup_{x,\, s,\, t} \Pr\{s, x, t, V_\varepsilon(x)\} = o(1), \quad h \downarrow 0,$$

where $x \in X$; s, $t \in [0, c]$, $0 < t - s \leqq h$ and $V_\varepsilon(x) = \{y : \rho(x, y) \geqq \varepsilon\}$. Then almost all sample functions of Markoff process x_t are continuous if and only if for each $\varepsilon > 0$

$$\int_0^{c-h} \Pr\{\rho(x_t, x_{t+h}) > \varepsilon\} dt = o(h), \quad h \downarrow 0.$$

Almost all sample functions of a Martingale (semi-martingale) x_t are continuous if and only if for $h \downarrow 0$

$$\int_0^{c-h} \Pr\{x_t < a, x_{t+h} > b\} dt = o(h),$$

$$\int_0^{c-h} \Pr\{x_t > b, x_{t+h} < a\} dt = o(h)$$

for each a and b, $(a < b)$.

(L. V. Seregin)

2/581

SEGUCHI, T. (Kyushu University, Japan) **10.1** (8.9)

Some statistical process controls. I. Controls of additive processes, II. Controls of finite Markoff chains—*In English*

Mem. Fac. Sci. A, **Kyushu** (1960) **14**, 139-178 (11 references)

In part I of this paper which occurs between pages 139 and 153, the author considers the controlling system of a discrete time parameter additive process. The system works to bring the process back to the reference value when the process shows a deviation larger than a preassigned constant from the reference value. The action of the system may contain a random error which is independent of all the other variables. The author gives the stationary distribution of the Markoff process representing the process under the control by this system: this section includes six references. In part II which follows (on pages 154 to 178) is given a controlling system for a Markoff chain. The system is supposed to work in such a way as to make the chain return to some preassigned state when it reaches some unfavourable state. After the detailed descriptions of the necessary quantities, the author gives a proof of the fact that the best controlling system or the system which gives the least stationary probability of the process being at some unfavourable state, is the one that makes the process return to the state which is the farthest, in the sense of the mean-length of the first passage time, from the boundary or the set of unfavourable states.

(H. Akaike)

2/582

SOEST, J. L. van (Technical University, Delft, Netherlands) **10.0** (10.5)

Order and disorder—*In English*

Statist. Neerlandica (1960) **14**, 249-258 (15 references)

The degree of order and of disorder are defined such that they are complementary and, both, vary between 0 and 1. Examples are given in problems of human life, and of human community with anarchistic and totalitarian extremes; also in thermo-dynamic entropy and information theory. In the latter problems the statistical quantity $\sum_{i=1}^{n} p(i) \log p(i)$ appears; it has quite the same meaning in both cases: the quantity of statistically weighted alternatives.

The relation to entropy and to quantity of selective information is discussed. The degree of order 1, degree of disorder 0, appears if this statistical quantity is 0, one of the $p(i)$ being equal to 1: " frozen " or " pure " state. The degree of order 0, degree of disorder 1, appears if this quantity is $\log n$, all $p(i)$ being equal to $1/n$. The degree of disorder is equal to $\sum_{i=1}^{n} p(i)^n \log p(i)$.

In information theory the degree of order is identical with redundancy. Order/disorder problems in patterns are related to Markoff chains of higher order and are not discussed here.

(J. L. van Soest)

2/583

STÖRMER, H. (Siemens u. Halske A.G., München) **10.4** (–.–)

A queueing problem in telephone exchanges—*In German*

Zeit. angew. Math. Mech. (1960) **40**, 236-246 (3 references)

The well-known distribution, first given by Erlang, of waiting times occurring when the central steering mechanism is working with a constant holding time T_0, applies to the case where the calls are handled in the order in which they arrive. In practice, this condition is very often not fulfilled. Instead, the waiting calls are dealt with according to chance.

In the present investigation the distribution of waiting calls is first treated under the assumption that the calls are being handled according to a system of priorities possessing m different steps. It is then assumed that there is a chance correlation between any one waiting call and one of these priority steps. In the limit $m \to \infty$ a simple closed expression is obtained for the distribution of the waiting times.

(H. Störmer)

2/584

STAM, A. J. (Physical Lab., Gov. Def. Org., The Hague, Netherlands) **10.0** (10.5)

Some mathematical remarks on information theory—*In Dutch*

Statist. Neerlandica (1960) **14**, 259-265 (18 references)

This paper is the elaboration of a lecture given recently by the author. It contains definitions of entropy and the amount of information as given by Shannon; some remarks on their interpretation and their generalisations to abstract probability spaces. Some remarks are also made on the mathematical problems arising from these definitions. Shannon's definition is compared with the concept of " intrinsic accuracy " or information used in statistics that was defined by Fisher. Some properties which these different kinds of " information " have in common are mentioned.

(A. J. Stam)

2/585

THEODORESCU, R. (Institute of Mathematics, Bucharest) **10.1** (10.8)

Multiple Markoff processes—*In Russian*

Rev. Math. Pures Appl. (1960) **5**, 341-362 (14 references)

In this paper a direct study of multiple Markoff processes is suggested by means of integro-differential equations. Starting from the functional equations of a multiple Markoff process [see Onicescu, *Probability theory* (1956) Bucuresti: Ed. Tehnică] which are replacing the Chapman-Kolmogoroff equation, in section one the corresponding differential equations are deduced for the case of a finite set of states. These equations are analogous to those given by Kolmogoroff for simple Markoff processes.

In section two multiple Markoff processes on the real axis are considered; the corresponding integro-differential equations are deduced and certain ergodicity properties are investigated. Further, conditions for existence and unicity for the corresponding solutions are provided.

Finally, the case leading to systems with partial derivatives similar to the Fokker-Planck equation, are also investigated.

The present paper represents a shortened Russian version of a sequel of papers published previously in Rumanian [*Bul. ştiinţific*, secţ. şt. mat. fiz. (1955) **7**, 763-774 and 776-794; *Analele Univ. C. I. Parhon*, ser. şt. naturii (1956) **10**, 23-24].

(M. Iosifescu)

2/586

THEODORESCU, R. (Institute of Mathematics, Bucharest) **10.1** (10.8)

Certain stochastic models for learning and their interpretation. Linear chains—*In French*

Second Hungarian Mathematical Congress (1961) (6 references)

A systematic treatment of stochastic models occurring in learning theory was given by Bush & Mosteller in their monograph [*Stochastic models for learning* (1955)].

In the present note it is shown that certain stochastic models for learning, considered by these authors, may be obtained by means of schemes involving chains with complete connections, introduced by Onicescu & Mihoc [see *Bull. Sci. Math.* (1935) **59**, 174-192]. A recent monograph deals with the detailed treatment of these processes [see Ciucu & Theodorescu *Processes with complete connections* (1960) Bucharest: Ed. Acad. R.P.R., in Rumanian].

(R. Theodorescu)

2/587

VOLKONSKY, V. A. (Moscow) **10.1** (–.–)

Construction of inhomogeneous Markoff processes by means of a random time substitution—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 47-65 (6 references)

It is proved by the author in this paper that a continuous single dimensional Markoff process $y(t)$ with wide restrictions can be obtained from the Wiener process $x(t)$ in the following form: $y(t) = \psi[x(\tau_t), t]$, where $\psi(x, t)$ is a continuous function, monotonic in x for a given t, and τ_t is a nondecreasing random function of t (theorem 1).

The author gives conditions which should be met by the Markoff process $x(t)$ in abstract space and the random function τ_t so that the process $y(t) = x(\tau_t)$ will also be a Markoff process (theorem 2).

(V. A. Volkonsky)

2/588

WIGGINS, A. D. (Hanford Laboratories Operation, G.E.C., Richland, Washington) **10.2** (10.9)

On a multicompartment migration model with chronic feeding—*In English*

Biometrics (1960) **16**, 642-658 (6 references)

A biological system with K compartments is considered, and the probability of a particle migrating from compartment i to compartment j in a small time, h, is $\lambda_{ij}h+0(h)$, where the λ_{ij} are constant migration rates.

A set of simultaneous differential-difference equations is derived for the transition probability of a particle starting in compartment $i = 1$ having migrated to compartment j at time t. When particles are introduced into the first compartment according to a known "feeding" function, the state of the system is described by the number of particles in the various compartments. By using generating functions, the stochastic behaviour is found in terms of the single-particle process. In particular, it is shown that the expected numbers of particles in the compartment satisfy the same set of equations as the original probabilities, except for the addition of non-homogeneous terms.

If experimental determinations of the numbers of particles in each compartment are made at discrete times, these may be used to estimate the migration rates λ_{ij}. The expectations and their time derivatives are replaced by the observations and their time differences in the set of equations, and a method similar to least-squares autoregression fitting is used to obtain the estimates. To facilitate the solution, an isomorphism is constructed between certain matrices of a block-pattern type and matrices of a lower order; multiplication of a matrix of the first type by one of the second is defined. The inversion of a block matrix is required for the solution, and this is achieved by proving that each block is a cyclic matrix whose inverse is therefore a power of itself.

Three unsolved problems are posed. In view of the randomness of the observations, do the linear estimating equations have a solution with probability one? What are the sampling properties of the estimators? Would their accuracy be greater if the observations were spaced at shorter rather than longer time intervals? On the basis of an artificial example with known migration rates, it is suggested that the answer to the third problem is in the affirmative.

(G. A. Watterson)

2/589

WILKINS, C. A. (University of New South Wales) **10.4** (–.–)

On two queues in parallel—*In English*

Biometrika (1960) **47**, 198-199 (1 reference)

In a recent paper Haight, F. A. [*Biometrika* (1958) **45**, 401-410; abstracted in this journal No. **1**/128, **10.4**] investigated a two-queue system in which an arrival is assigned to the shorter queue, or if they are of equal length, to a specified one, the "nearer" one.

The results given by Haight are here briefly extended to the general case where, if X is the length of the near queue, Y the length of the other, and $W(XY)$ the probability of an arrival joining the near queue,

(i) $W(x, y) = 1, x<y$

(ii) $W(x, y) = w(x), x = y$

(iii) $W(x, y) = 0, x>y.$

The modifications of Haight's results are set out by the present author.

(Florence N. David)

2/590

YEO, G. F. (University of Western Australia) **10.4** (-.-)

The time-dependent solution for a dam with geometric inputs—*In English*

Aust. J. Appl. Sci. (1960) **11**, 434-442 (6 references)

Let the content $Z_t = 0, 1, 2, \ldots$, of an infinite dam at the discrete times $t = 0, 1, 2, \ldots$, be defined by

$$Z_{t+1} = Z_t + X_t - \min(Z_t + X_t, 1),$$

where $X_t = 0, 1, 2, \ldots$, is a random additive input during the interval $(t, t+1)$; the process Z_t is known to be a Markoff chain. This paper is concerned with the explicit time-dependent solution for the probability

$$P_i(t) = \Pr(Z_t = i \mid Z_0 = u); \quad i = 0, 1, 2, \ldots,$$

the initial content u being a non-negative integer when X_t follows a geometric distribution.

Using occupancy methods, the probability $P_0(t)$ of emptiness of the dam is first found. The generating function $P(s, t) = \sum_{i=0}^{\infty} P_i(t)s^i$ $(|s| \leqq 1)$ is shown to depend on $P_0(t)$, and the author makes use of residual methods to obtain explicit expressions for the $P_i(t)$ from it.

The stationary probabilities $P_i = \lim_{t \to \infty} P_i(t)$ of the process give results previously obtained by Prabhu [*Ann. Math. Statist.* (1958) **29**, 1234-1243; abstracted in this journal No. **1**/310, **10.4**].

(J. Gani).

2/591

ZÍTEK, F. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **10.0** (0.5)

The theorem of Saks concerning the functions of random intervals—*In French*

Second Hungarian Mathematical Congress (1961) (1 reference)

In the theory of the Burkill integral a theorem of Saks is known stating approximately the following: if f is an integrable, in the sense of Burkill, and derivable interval function; then its integral is also derivable and the two derivatives are almost everywhere equal.

The aim of the present note is to show that an analogous theorem holds for the functions of random intervals. The theory of these functions and of the (BB) integral, which is a generalisation of the Burkill integral, was developed in detail in the author's papers [*Czech. Math. J.* (1958) **8**, 583-609; abstracted in this journal No. **1**/150, **10.0**] and a further paper in the press. See also *Cas. pěst. math.* (1959) **9**, 83-89; abstracted in this journal No. **1**/321, **10.0**].

(F. Zítek)

2/592

HOSTELET, G. (41 Avenue W. Churchill, Bruxelles) **11.0** (0.0)

The co-operation of mathematics in the measurement and interpretation of characteristics of statistical phenomenae—*In French*

Bull. Int. Statist. Inst. (1960) **37**, III 375-384

Mathematical statistics mean the co-operation of mathematics to investigate the measurement and the experimental analysis of specific statistical groups. An explanation is given of four out of the six main tasks of mathematical statistics with special emphasis on the relations between mathematical and experimental analysis.

In the first part the empirical character of the algebraic fitting of frequency polygons or of time series is emphasised. The second step consists of obtaining, with the help of the algebraic smoothing formula, the algebraic expressions of significant characteristics of a statistical feature, where these characteristics had beforehand been defined by experimental analysis. The quantitative conditions of the validity of the experimental signification of these algebraic expressions are explained.

The third step is the discovery of new characteristics by the experimentally explanatory analysis of the equation of the frequency curve or time series. Here, once more, the quantitative conditions of the experimental validity of new characteristics are explained. The other tasks imply the application of lottery systems: the author states that he has discriminated between two kinds of probabilities:

(i) the probability of the existence or the production of a statistical fact;
(ii) the probability of the exactitude of the measurement of a statistical fact.

The fourth step is the search for measurements of observations which do not conform with the formulae of the lottery system, to which the total of observed statistical characteristics has been referred. It will thus be a matter of discovering the experimental significance of this disagreement.

It is known that a group of observations relating to a statistical feature, for which the number of units is less than that of another group, could provide an estimate closer to the characteristic sought than the one provided by the other group. And without it being known. In statistical observations only probable degrees of accuracy can, therefore, be established. In order to obtain this

continued

HOSTELET, G. (41 Avenue W. Churchill, Brussels) **11.0** (0.0) *continued*

The co-operation of mathematics in the measurement and interpretation of characteristics of statistical phenomenae—*In French*

Bull. Int. Statist. Inst. (1960) **37**, III 375-384

the n statistical observations are compared with n drawings in a lottery system of which the composition must be adapted to the studied data. Attention is drawn to the quantitative and qualitative conditions which ensure the experimental validity of the lottery system used and of the results derived from it. An example of the determination of significant characteristics in statistical observations is analysed.

The fifth task of mathematical statistics: the evaluation of probable degrees of approximation of calculated statistical characteristics; and the sixth: the working out of sampling methods have not yet been dealt with. The paper aims to explain the quantitative conditions of the experimental validity of formulae and equations relating to the measurement of socio-statistical data, and states that these conditions should be more often taken into consideration.

(G. Hostelet)

FREUDENTHAL, H. (Schubertstraat 44, Utrecht) **11.0** (11.9)

Order and disorder—*In Dutch*

Statist. Neerlandica (1960) **14**, 235-247

Order is a quite definite notion as long as it means a special kind of order. Disorder in the sense of absence of any order is a vague notion. Statistical notions do not depend on notions like order and disorder. Instead of disorder, independency is the fundamental statistical notion. Statistical independency can be obtained by creating disorder. But pseudo-random-material shows that it can also be obtained by suitable strictly causal methods. It is not possible to test whether a given material is produced by stochastic devices. It is dangerous to try disorder as a null-hypothesis in order to refute it by showing some regularities in the underlying material. This is shown by the analysis of statistical sham proofs for the impossibility of *generatio spontanea*, such as tried by some biologists.

(H. Freudenthal)

2/595

SMIRNOFF, S. V. & POTAPOFF, M. K. (Moscow) **11.1** (3.1)

Nomogram for probability functions χ^2—*In Russian*

Teor. Veroyat. Primen. (1961) **6**, 138-140 (3 references)

In this paper a nomogram is constructed for the function

$$\Pr(\chi^2, n) = \frac{1}{2^{(n-2)/2}\Gamma(\frac{1}{2}n)}\int_{\chi}^{\infty} z^{n-1}\, e^{-z^2/2}\, dz$$

or the variables P, χ^2, n lying within the following limits:

$$1 \leqq n \leqq 110,\ 1 \leqq \chi^2 \leqq 150,\ 0{\cdot}001 \leqq P \leqq 0{\cdot}999.$$

The relative error in the middle part of the answer scale of P does not exceed 3 per cent., for $0{\cdot}1 \leqq P \leqq 0{\cdot}9$, and 10 per cent. at the ends of this scale.

(S. V. Smirnoff)

2/596

NEYMAN, J. (University of California, Berkeley) **11.9** (10.0)

Indeterminism in science and new demands on Statisticians—*In English*

J. Amer. Statist. Ass. (1960) **55**, 625-639 (14 references, 3 figures)

The words " indeterministic study " are used to designate research aiming to determine how frequently a quantity X characterising the phenomena considered assumes its various particular values. If the purpose of research is to establish the exact value of X as a function of other variables, then this research is " deterministic ". In the history of indeterminism in science four overlapping periods are discernible.

(i) Period of " marginal indeterminism." This was the period, symbolised by the names of Laplace and Gauss, in which research in science was all deterministic with just one domain, that of errors of measurement, treated indeterministically.

(ii) Period of " static indeterminism," is symbolised by the names of Bruns, Charlier, Edgeworth, Galton and Karl Pearson. Here, the main subject of study was a "population" and efforts were made to develop systems of frequency curves to describe analytically the empirical distributions.

(iii) The third discernible period, may be termed the period of " static indeterministic experimentation." It is marked by the name of R. A. Fisher and by his book *The design of experiments*. The typical problems considered were: do these two populations have the same distributions of X? This and similar questions led to the development of basic ideas of tests of statistical hypotheses and of estimation, and also of the appropriate techniques. All of these are currently at the disposal and in constant use by an applied statistician.

2/597

(iv) The fourth period in the history of indeterminism, currently in full swing, the period of " dynamic indeterminism," is characterised by the search for evolutionary chance mechanisms capable of explaining the various frequencies observed in the development of the phenomena studied. The chance mechanism of carcinogenesis and the chance mechanism behind the varying properties of the comets in the Solar System exemplify the subjects of dynamic indeterministic studies. One might hazard the assertion that every serious contemporary study is a study of the chance mechanism behind some phenomena. The statistical and probabilistic tool in such studies is the theory of stochastic processes, now involving many unsolved problems. In order that the applied statistician be in a position to cooperate effectively with the modern experimental scientist, the theoretical equipment of the statistician must include familiarity and capability of dealing with stochastic processes. (J. Neyman)